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Properties of Absolute Stability in the Presence of Time Lags

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Abstract: This study is concerned with the properties of absolute stability independent of the delays of time-delay systems, possessing non commensurate internal point delays, for any nonlinearity satisfying a Popov's- type time positivity inequality. That property holds if an associate delay-free system is absolutely stable and the size of the delayed dynamics is sufficiently small. The results are obtained for nonlinearities belonging to sectors [0, k] and [h, k+h], $k \geq 0$ and are based on a parabola test type.

Key words: Absolute stability, dynamic systems, nonlinear feedback, time-delay systems

INTRODUCTION

The absolute stability of dynamic is an interesting issue since it refers to the global asymptotic stability of a system under any feedback law provided by a wide class of nonlinear devices. Such nonlinear devices have to satisfy a certain positivity sector-type constraints. The problem has been widely studied for the plant delay-free case and nonlinear feedback devices within linear sectors [k_1, K_2] and (k_1, K_2) in (0, ∞) [1-6]. Some of those results have been extended to single- delay cases provided that the transfer function of the linear subsystem is (non critically) stable (i.e., With poles in Re s < 0) provided that its H_∞-by is upper-bounded with a sufficiently small upper-bound and that the feedback nonlinear device satisfies certain local Liptschitzian [regularity conditions[7] and to systems with external delays (i.e., in the input)[8]. In this study, such assumptions are removed by allowing nonlinearities simply satisfying a (in general non symmetric) sector-type positivity constraints, multiple non commensurate internal (i.e., in the state) delays and either strictly stable (the so-called principal case) or critically stable (the so-called simplest particular case) linear plants with a single critically stable pole at s=0.

Notation:

* An output- feedback nonlinearity $\Phi(y(t))$ in a Popov´s sector $\left[k_1, k_2\right] \subset [0, \infty)$ means that the scalar real function Φ: R×[0, t] → R is such that $k_1 y(t) \leq \Phi(y(t)) \leq k_2 y(t)$ for all $t \geq 0$ with $\Phi(y(t)) = 0$ if and only if y(t)=0.
* The Hardy space RH_∞ of the matrices G (s) tions or matrices G (s) are proper real rational functions with all its poles in Re s < 0 (i.e., Strictly stable) and H_∞-norm $\left\| G(s) \right\|_\infty$ with $\mathbf{R}_0^+ = \mathbf{R}^+ \cup \{0\}$ and $\lambda_{Max}(.)$ being the maximum eigenvalue of the (.)- symmetric matrix.
* It is said that a transfer function G(s) or matrix is strictly stable if $G(s) \in RH_\infty$ and its characteristic polynomial (or quasi-polynomial in the presence of internal delays) is strictly Hurwitzian.
* A linear transfer function G(s) is in the principal case if it belongs to RH_∞ and its characteristic polynomial (or quasi-polynomial in the presence of internal delays) is strictly Hurwitzian. It is in the simplest particular case if $G(s) = \frac{G_0(s)}{s}$ with $G_0(s) \in RH_\infty$.
* An unforced linear system with r finite internal point delays h_i of state equation $\dot{x}(t) = Ax(t) + \sum_{i=1}^{r} A_i x(t - h_i)$ has two associated systems without delays, namely:

$\dot{z}_1(t) = \left(A + \sum_{i=1}^{r} A_i\right) z_1(t)$ Which describes the above so-called *current delay-free system* time-delay system when $h_i = 0;\ i = \overline{1, r}$; and $\dot{z}_2(t) = Az_2(t)$ Which is called the *nominal delay-free system* which describes the above time-delay system when $A_i = 0$, or when $h_i \to \infty;\ i = \overline{1, r}$.

Both systems have to be stable in order that the delay system is a stable independent of the delays:

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- * The l_2 - norm of a matrix (or vector) M is denoted as $\|M\|_2 = \lambda_{Max}^{1/2}\left(M^T M\right)$. In vectors such a norm coincides with the Euclidean norm.

Descriptions of time-delay systems under sector-type nonlinear feedback: Consider the single-input single-output linear and time-invariant system:

$$\dot{x}(t) = A\,x(t) + \delta\sum_{i=1}^{r} A_i\, x\left(t - h_i\right) + b\,u(t) \qquad (1.a)$$

$$y(t) = c^T x(t) + d\,\xi(t) \qquad (1.b)$$

Under a nonlinear output-feedback law:

$$u(t) = \xi(t) = -\Phi\left(y(t)\right) \qquad (1.c)$$

Or:

$$u(t) = \dot{\xi}(t) = -\Phi\left(y(t)\right) \qquad (1.d)$$

where, $x(t)\in \mathbf{R}^n$, $u(t)\in \mathbf{R}$, $y(t)\in \mathbf{R}$ are the state, input and output, respectively, and A, A_i ; $i = \overline{1,\ r}$,are real square n- matrices, $b, c \in \mathbf{R}^n$, $d \in \mathbf{R}$ and δ is a real scalar parameter which is introduced by convenience to govern the size of the delayed dynamics for given matrices A_i ; $i = \overline{1,\ r}$. The initial condition of (1.a) is any absolutely continuous function $\varphi : [-h, 0] \to R^n$ plus, eventually, a function of zero measure of isolated bounded discontinuities defined on $[-h, 0]$ where $h = \underset{1\le i\le r}{Max}\left(h_i\right)$. The nonlinear feedback device is defined via (1.c) or (1.d) by a nonlinear function $\Phi : \mathbf{R}\times[0, t] \to \mathbf{R}$ satisfying $\Phi(y)=0$ if and only if y = 0 and $k_1 y \le \Phi(y) \le k_2 y$. The feedback configuration (1.a) - (1.c) is called the principal case and (1.a) - (1.b) and (1.d) is called the simplest particular case, both satisfying that the roots of $\mathrm{Det}\left(sI - A - \sum_{i=1}^{r} A_i\, e^{-h_i s}\right) = 0$ implied Re s < 0. In the second situation, the linear device adds a critically stable simple pole at s =0. The transfer function of (1) becomes:

$$G(s) = \frac{M(s)}{N(s)} = c^T\left(sI - A - \delta\sum_{i=1}^{r} A_i\, e^{-h_i s}\right)^{-1} b + d_0(s) \qquad (2)$$

With $d_0(s) = d$ (Principal case), $d_0(s) = d/s$ (Simplest particular case). Direct calculations with (2) yield:

$$G(s) = c^T\left[\left(sI - A\right)\left(I - \delta\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r} A_i e^{-h_i s}\right)\right)\right]^{-1} b + d_0(s) \qquad (3)$$

Note that the identity

$$\left[\left(I - \delta\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r} A_i\, e^{-h_i s}\right)\right)\right]^{-1} = I + \Delta\left(s, \delta\right)\left(sI - A\right) \qquad (4)$$

Holds provided that the inverse exists for Re s $\ge$ 0 with $\Delta(s, \delta)$ being defined by:

$$\Delta(s, \delta) = \delta\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r} A_i e^{-h_i s}\right)\left(sI - A - \delta\sum_{i=1}^{r} A_i\, e^{-h_i s}\right)^{-1} \qquad (5)$$

What follows directly since from direct calculations:

$$\left(I + \Delta(s, \delta)(sI - A)\right)\left(I - \delta\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r} A_i e^{-h_i s}\right)\right) = I$$

The substitution of (4) into (3) yields:

$$G(s) = G_0(s) + c^T \Delta\left(s, \delta\right) b \qquad (6)$$

where, $G_0(s) = c^T\left(sI - A\right)^{-1} b + d_0(s)$ is the transfer function of the nominal delay-free system. The following result holds trivially for the existence of the inverse in (4) for Re s $\ge$ 0 if $G_0 \in RH_\infty$, namely, if the nominal delay-free system is (non critically) stable.

Proposition 1: The linear forward loop of system (1) is (non critically) stable independent of the delays if A is a stability matrix (or, equivalently, if $G_0 \in RH_\infty$) and $|\delta| < \delta_0^{-1}$ with $\delta_0 = \left\|\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r} A_i\, e^{-h_i s}\right)\right\|_\infty$ what is guaranteed if $|\delta| < \varepsilon_0^{-1}$, with $\varepsilon_0 = \varepsilon_1\left(\sum_{i=1}^{r} \|A_i\|_2\right) \ge \delta_0$,and $\varepsilon_1 = \left\|\left(sI - A\right)^{-1}\right\|_\infty$. Furthermore:

$$\|G\|_\infty \le \|G_0\|_\infty + \|c\|_2\|b\|_2\,\bar{\varepsilon}(\delta)$$
$$\le \|G_0\|_\infty + \|c\|_2\|b\|_2\,\varepsilon(\delta) \quad (7)$$

the first inequality holding if $|\delta| < \delta_0^{-1}$ and both inequalities holding if $|\delta| < \varepsilon_0^{-1}$ where:

$$\bar{\varepsilon}(\delta) = \frac{\delta\,\delta_0\,\varepsilon_1}{1-|\delta|\,\delta_0} \le \varepsilon(\delta) = \frac{\delta\varepsilon_0\varepsilon_1}{1-|\delta|\varepsilon_0} \quad (8)$$

Note that (1.a) may be equivalently rewritten as:

$$\dot{x}(t) = A'x(t) + \sum_{i=1}^{r} A_i\left(x\left(t-h_i\right) - x(t)\right) + bu(t) \quad (9)$$

where, $A' = A + \sum_{i=1}^{r} A_i$ is the matrix of dynamics associated with the current delay-free system describing (1.a) for $h_{(.)} = 0$ so that it should be a stability matrix in order that (1) be stable independent of the delays. Thus, (6) may be rewritten equivalently as:

$$G(s) = c^T\left(I + \Delta'(s,\delta)(sI - A')\right)^{-1}\left(sI - A'\right)^{-1} b + d_0(s)$$
$$= G_0(s) + c^T\Delta'(s,\delta)b \quad (10)$$

where $G_0(s) = c^T\left(sI - A'\right)^{-1} b + d_0(s)$, and:

$$\Delta'(s,\delta) = \delta\left(sI - A'\right)^{-1}\left(\sum_{i=1}^{r} A_i\left(e^{-h_i s} - 1\right)\right)$$
$$\left(sI - A' - \delta\sum_{i=1}^{r} A_i\left(e^{-h_i s} - 1\right)\right)^{-1} \quad (11)$$

Thus, one gets a subsequent parallel result of Proposition 1 since $G_0' \in RH_\infty$ implies, and it is implied since G_0' is properly, that both A' and A are stability matrices and $\sup_{\omega\in\mathbf{R}}\left(\left|e^{-j\omega h_i} - 1\right|\right) = 2$.

Proposition 2: Assume that $G_0' \in RH_\infty$. The linear forward loop of system (1) is (non critically) stable independent of the delays if A' is a stability matrix and $|\delta| < 1/\delta_0'$ with:

$$\delta_0' = \left\|\left(sI - A'\right)^{-1}\left(\sum_{i=1}^{r} A_i\left(e^{-h_i s} - 1\right)\right)\right\|_\infty \quad (12.a)$$

what is guaranteed if $|\delta| < 1/\varepsilon_0'$, with:

$$\varepsilon_0' = 2\varepsilon_1'\left(\sum_{i=1}^{r}\|A_i\|_2\right) \ge \delta_0'$$
$$\varepsilon_1' = \left\|\left(sI - A'\right)^{-1}\right\|_\infty \quad (12.b)$$

Furthermore:

$$\|G\|_\infty \le \|G_0'\|_\infty + \|b\|_2\|c\|_2\,\bar{\varepsilon}(\delta)$$
$$\le \|G_0'\|_\infty + \|b\|_2\|c\|_2\,\varepsilon'(\delta) \quad (13)$$

the first inequality holding if $|\delta| < \delta_0^{-1}$ and both inequalities holding if $|\delta| < \varepsilon_0^{-1}$ where:

$$\bar{\varepsilon}'(\delta) = \frac{\delta\,\delta_0'\,\varepsilon_1'}{1-|\delta|\,\delta_0'} \le \varepsilon'(\delta) = \frac{\delta\,\varepsilon_0'\,\varepsilon_1'}{1-|\delta|\,\varepsilon_0'} \quad (14)$$

Remarks 1: Note that the definition of H_∞ - norms for matrices implies:

$$\left\|\left(sI - A\right)^{-1}\left(\sum_{i=1}^{r}\|A_i\|_2\right)\right\|_\infty \le \left\|\left(sI - A\right)^{-1}\right\|_\infty\left(\sum_{i=1}^{r}\|A_i\|_2\right)$$

and a similar expression for the replacement of A by A'.

Note that in the simplest particular case, Propositions 1-2 do not hold since the delay- free linear system has a critically stable pole. However, both Propositions hold for nominal and current delay-free subsystems described by transfer functions $G_0^*(s)$ and $G_0^{*'}(s)$ being strictly stable such that $G_0(s) = s\,G_0^*(s)$ and $G_0'(s) = sG_0^{*'}(s)$.

Popov´s parabola tests of time-delay systems with point delays

Popov´s parabola tests on sectors $[0,k]$ for the Principal case and $(0,k)$ for the simpleof particular case: Note that the results for the linear part of the delayed system are analyzed together with the nonlinear feedback law (1.c) t, establish the causeabola te, ts on sectors [0,k] (principal case) and (0,k) (Simplest particular case). It is well known for the delay-free case[2,9,10] and for the case of presence on external delays only[8], that positive parabola tests guarantee absolute stability. In Section 4, the absolute stability problem is extended for systems with internal point delays based on parabola tests which are now addressed. The amount of tolerance to the delayed dynamics is made explicit so that the parabola test is positive on a Popov´s sector [0, k] provided that it is positive on a sector [0, k_0] for the Principal Case. The reasoning guidelines are similar fort the simplest particular case on the respective sectors (0, k) and (0, k_0). The decomposition of the whole transfer function G(s) as in (6) subject to (5) by using the nominal delay-free system transfer function $G_0(s)$ or using (10)-(11) with that of the current delay-free system $G_0{}'(s)$ is used to obtain the subsequent result.

Proposition 3: Assume that there is a real constant $q_0>0$ such that $\mathrm{Re}\left((1+q_0\omega)G_0(j\omega)+\frac{1}{k_0}\right)>0$ for some finite real constants q_0 and k_0. Thus, $\mathrm{Re}\left((1+q\,\omega)G\,(j\omega)+\frac{1}{k}\right)>0$ for any real constants q and k satisfying $0<q\le q_0$ and $0<k<k_0$ provided that:

$$|\delta|<\mathrm{Min}\left(\frac{1}{\varepsilon_0},\ \frac{(1-\varepsilon_0)\ (k_0-k)}{k_0\,k\,(1+q\,k_1)\,\|b\|_2\,\|c\|_2\,\varepsilon_0\varepsilon_1}\right)$$

With the real constants $\varepsilon_{0,1}$ defined as in Proposition 1 and k_1 being a finite real constant satisfying:

$$k_1\ge\frac{\|s\,\tilde{G}(s)\|_\infty}{q\|\tilde{G}(s)\|_\infty}=\frac{\|s\,(G(s)-G_0(s))\|_\infty}{q\,\|(G(s)-G_0(s))\|_\infty}$$

Remark 3: Proposition 3 applies to both the principal and simplest particular cases. However, for the principal case, closed sectors [0, k_0] and [0, k] may be considered with k_0 being finite or infinity and $q\le q_0$ for any real constant q_0. The proof is direct from previous results for the undelayed case[2,3,9,10].

Popov´s parabola tests on sectors $[h,k+h]$ and $(h,k+h)$: The absolute stability on sectors $[k,k+h]$ for the Principal Case and (k, k+h) for the Simplest Particular one may be performed equivalently via the use of the modified transfer function:

$$G_m(s)=G_{0m}(s)+\tilde{G}_m(s)=\frac{G_0(s)+\tilde{G}(s)}{1+h\left(G_0(s)+\tilde{G}(s)\right)}$$

checked for absolute stability on closed or open sectors [0, k] or (0,k) with the current nominal delay-free transfer function $G_{0m}(s)=\frac{G_0(s)}{1+h_0G_0(s)}$ being checked on respective nominal sectors $[0,k_0]$ or $(0,k_0)$ for $h=h_0+\Delta h$ and $\tilde{G}(s)=c^T\Delta(s,\delta)b$ includes the effects of the delayed dynamics. Now, if $G_{0m}(s)$ is absolutely stable for nonlinearities in $[0,k_0]$ (or in $(0,k_0)$), thus, $G_m(s)$ is absolutely stable in $[0,k_0+h]$ (or in $(0,k_0+h)$) for $\Delta h=0$ and $\tilde{G}(s)=0$; i.e., in the absence of delayed dynamics provided that the tested Popov´s sector is not modified. Now, the basic idea to be developed in the following is summarized as follows. Assume that $G_{0m}(s)$ is absolutely stable in a reference nominal sector. Calculate a sector potential modification (in terms of a maximum allowable $k<k_0$ and $|\Delta h|\ge 0$) and a tolerance to delayed dynamics (in terms of maximum allowable $|\delta|>0$ to quantify a maximum allowable $\|\tilde{G}(s)\|_\infty$ for give $A_{(.)}$-matrices) such that the current system involving delays remains absolutely stable independent of the sizes of the delays. The subsequent result is an extension of Proposition 2.

Proposition 4: Assume that there is a real constant $q_0>0$ such that:

$\mathrm{Re}\left[\,(1+q_0\,\omega)\,G_{0m}(j\omega)+k_0^{-1}\right]>0$ for some finite real constant $k_0>0$. Thus, $\mathrm{Re}\left[\,(1+q\,\omega)\,G_m(j\omega)+k^{-1}\right]>0$ independent of the delays for any real constants q and k satisfying $0<q\le q_0$ and $0<k<k_0$ provided that $|\delta|$ and $|\Delta h|$ are sufficiently small.

The links between the above parabola tests and absolute stability of time-delay systems independent of the delays are given in the subsequent section.

Absolute stability: In this section, it is proved that the parabola tests of Propositions 3-4 of Section 3 and its modifications based on the use of the current delay-free system (see, for instance, Remark 3) guarantee the absolute stability of (1. a) -(1.b) for all nonlinear feedback law (1.c) or (1.d) in the corresponding Popov´s sector. To address the intended result, it is proved that a quality measure of the input-output energy time-integral of the linear forward loop is bounded for all time under any feedback law of the given class if the related parabola test is positive. The output (1.b) may be decomposed as the sum of the unforced response plus the forced one as follows:

$$y(t) = y_{uf}(t) + y_f(t) \quad (15)$$

$$y_{uf}(t) = c^T \Psi(t)\bar{x}_0 =$$

$$c^T\left(\Psi(t)x_0 + \int_0^t \sum_{i=1}^r \Psi(t-\tau-h_i)\varphi(\tau)d\tau\right) \quad (16.a)$$

$$y_f(t) = c^T\int_0^t \Psi(t-\tau)\,b\,u(\tau)d\tau + d\,\xi(t) \quad (16.b)$$

for all $t \geq 0$, where:

$$\bar{x}_0 = x_0 + \int_0^t \sum_{i=1}^r \Psi(h_i - \tau)\varphi(\tau)d\tau, \quad y_{uf}(t)$$

and $y_f(t)$ are the unforced and forced output responses of (1) where $x_0 = \varphi(0)$ and $\Psi(t)$ is the fundamental matrix of (1.a) which satisfies:

$$\dot{\Psi}(t) = A\Psi(t) + \sum_{i=1}^r A_i\Psi(t-h_i) \quad (17)$$

where, $\Psi(0) = I$ and Ψ (t)=0 for t< 0. If Proposition 3 holds then the stability of (1. a) -(1.b) is guaranteed. Thus, $\sup_{t\geq 0}\left(\|\Psi(t)\|_2\right) \leq k_\Psi < \infty$ and from (15) and (16.a):

$$|y(t)| \leq |y_{uf}(t) + y_f(t)|$$
$$\leq k_\Psi \|\bar{x}_0\|_2 \|c\|_2 + |y_f(t)| \quad (18)$$

since $\Phi(y(\tau))y(\tau) \geq 0$ for $\Phi(.)$ belonging to a Popov´s sector like those addressed in Section 3. This leads directly to:

$$\int_{t_0}^t y(\tau)u(\tau)d\tau = \int_{t_0}^t \left(y_f(\tau) + y_{uf}(\tau)\right)d\tau \leq \gamma_0^2 \quad (19)$$

what implies from (18):

$$\int_{t_0}^t y_f(\tau)u(\tau)d\tau \leq \gamma_0^2 + k_\psi \|\bar{x}_0\|_2 \|c\|_2 \int_{t_0}^t |u(\tau)|d\tau \quad (20)$$

Note that the Fourier transforms of u (t) and y (t) (denoted with capital letters) fulfill the frequency domain relation $Y(j\omega)U(-j\omega) = G(j\omega)|U(j\omega)|^2$ for all frequency ω where G (jω) is the f1. q) ency response of (1.a)-(1.b). Define for any scalar or vector signal f (t), a related signal $f_t(\tau) = f(\tau)$ for all real $\tau \in [0, t]$ and $f_t(\tau) = 0$ otherwise in $(-\infty, \infty)$ and any t ≥ 0. Its Fourier transform, which always exists for all finite t and also as $t \to \infty$ if f (t) is absolutely integrable on $(-\infty, \infty)$, is denoted by $F_t(j\omega)$. Since $u(t) = -\Phi(t)$, $-k \leq -(u(t)/y(t)) \leq k$ for all time, what implies from (18) and the use of Parseval' s theorem in (20):

$$\int_{-\infty}^{\infty} G(j\omega)|U(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} Y_f(j\omega)U(-j\omega)d\omega$$
$$\leq 2\pi\left(\gamma_0^2 + k_\Psi \|\bar{x}_0\|_2 \|c\|_2 \int_{-\infty}^{\infty} |u_t(\tau)|d\tau\right) \quad (21)$$

Define $\hat{G}(j\omega) = (1 + j\omega q)G(j\omega)$ and $\hat{U}(j\omega) = \frac{1}{1+j\omega q}|U(j\omega)|^2$ is the Fourier' s transform of:

$$\hat{u}(t) = q^{-1}e^{-q^{-1}t}\left[\hat{x}_u(0) + \int_0^t e^{-q^{-1}\tau}u^2(\tau)d\tau\right]$$

where, $\hat{x}_u(0)$ is the initial condition of the first-order filter $\frac{1}{1+qs}$. Thus, one gets:

$$\int_{-\infty}^{\infty} y_t(\tau)u_t(\tau)d\tau \geq -\frac{1}{k}\int_{-\infty}^{\infty} \hat{u}_t^2(\tau)d\tau - \gamma_1 \quad (22)$$

For some real constant $\gamma_1 > 0$ since $\hat{u}(t)$ is the output of a first-order exponentially stable filter of inpuand existand since the Fourier transforms $\hat{U}_t(j\omega)$ and $\hat{Y}_{ft}(j\omega)$ exist for any finite t, the substitution of (22) and (21) into (20) yields:

$$2\pi\left(\gamma_0^2 + \hat{k}k_\Psi \|\bar{x}_0\|_2 \|c\|_2 \int_{-\infty}^{\infty} |\hat{u}_t(\tau)|d\tau\right) + \gamma_1$$
$$\geq \min_{\omega \in R_0^+}\left(\mathrm{Re}\hat{G}_t(j\omega) + k^{-1}\right)\int_{-\infty}^{\infty} \hat{U}_t(j\omega)d\omega$$
$$= \min_{\omega \in R_0^+}\left(\mathrm{Re}\,\hat{G}(j\omega) + k^{-1}\right)\int_{-\infty}^{\infty} \frac{|U_t(j\omega)|^2}{1+j\omega q}d\omega$$
$$= 2\pi \min_{\omega \in R_0^+}\left(\mathrm{Re}\hat{G}(j\omega) + k^{-1}\right)\int_{-\infty}^{\infty} \hat{u}_t^2(\tau)d\tau \quad (23)$$

since the hodo graph G (jω) is symmetric with respect to the real axis of the complex plane, Re G (jω)= Re G (-jω) and Im G (jω)= - Im G (-jω) ($\Rightarrow \int_{-\infty}^{\infty} \mathrm{Im}\,\hat{G}(j\omega)d\omega = 0$) for all frequencies so that it suffices to test $\mathrm{Re}\,\hat{G}(j\omega)$ for $\omega \in R_0^+$. By taking any sufficiently large t=T in (28), it follows that $\hat{u}_t(\tau)$ is uniformly bounded for all $\tau \in [0, t]$, $t \geq T$ and of absolute value which can be equal to or larger than unity only over a subset of [0,t] of finite measure since $\hat{u}_t(.)$ is a continuous function which is the output of a stable filter if q>0. By taking $t \to \infty$, one concludes from (1.c) - (1.d) that the signal u (t), y (t), $\hat{u}_t(.)$, $\hat{y}_t(.)$ cannot diverge on $[0, \infty)$ for the linear plant (1.a) - (1.b) being in the principal case provided that G (s) has no unstable zero/pole cancellation provided that the inequality of Proposition 3 (see also the variant of Remark 4) holds. In the simplest particular case, the conclusion is identical by extending slightly the above reasoning if the inequality of Proposition 4 or its subsequent extension hold. Assume that $\hat{u}_t(.)$ diverges as $t \to \infty$. Thus, its squared value diverges at a slower rate from (28) except if u (t) is zero over a real set of infinite measure. In that case, Φ(.) and then y (.) would be zero over a set of infinite measure as well. Since y (.) is a continuous function then $y(t) \to 0$ as $t \to \infty$ while it is bounded. Thus, $u(t) \to 0$ as $t \to \infty$ while being bounded provided that G(s) has no unstable zero/pole cancellation. The final conclusion is that the system is globally Lyapunov´s stable for all nonlinearity in the appropriate closed (Principal Case) or open (Simplest Particular Case) Popov´s sector under feedback (1.c) or (1.d). It remains to be proved the global Lyapunov´s asymptotic stability of (1. a) -(1.c) and (1. a) -(1-b) and (1.d) to conclude the absolute stability of those feedback configurations. Since u (t) is bounded on $[0, \infty)$, then $\hat{u}_t(\tau)$ from its definition is absolutely integrable on $[0, \infty)$. Thus, the first term of (23) is bonded so that after comparison with its last term, it follows that the input converges asymptotically to zero if Proposition 3 or Proposition 4 hold. Thus, the absolute stability has been proved what is summarized in the subsequent main results under some of the following Assumptions:

Assumption 1: The pair (A, b) is controllable and the pair (A, c) is observable.

Assumption 1: The pair (A', b) is controllable and the pair (A', c) is observable.

Assumption 2: The pair (A', b) is stabilizable and the pair (A', c) is detectable.

Theorem 1: Assume that either Proposition 3 holds or, alternatively, the constraint of Remark 4 holds for q>0 ($q \geq 0$ for the Principal Case). Thus, any minimal realization (1. a) -(1.b) is absolutely stable (i.e. global asymptotically Lyapunov´s stable) under feedback (1.c) [Principal Case] for all Φ (.) in (0, k) -or (1.d) [Simplest Particular Case] for all Φ (.) in (0, k). As a result, $u: [0, \infty) \to R$, $y: [0, \infty) \to R$ and $\Phi: R \times [0, \infty) \to R$ are bonded and converge asymptotically to zero. That result also holds for nonlinearities Φ (.) in sectors [h, k+h] for the Principal Case and (h, k+h) for the Simplest Particular Case if the requirement of fulfillment of Proposition 3 is replaced for that of Proposition 4.

Direct extensions of Theorem 1 concerning the kind of state-space realizations (1. a) -(1.b) lead to the subsequent results.

Corollary 1: Proposition 3 (or, alternatively, the constraint of Remark 4) guarantee for q>0 that any stabilizable and detectable non minimal realization of G(s) is absolutely stable for Φ (.) in [0, k] (Principal Case) or (0, k) (Simplest Particular Case). That result holds for Φ (.) in [h, k+h] (Principal Case) or (h, k+h) (Simplest Particular Case) if the requirement of fulfillment of Proposition 3 is replaced for that of Proposition 4.

Corollary 2: Propositions 3 (or, alternatively, the constraint of Remark 4) and Proposition 4 guarantee for q>0 that any stable non minimal realization of G(s) (even not being stabilizable/ detectable) is absolutely stable for the same corresponding Popov´s sectors referred to in Theorem 1 and Corollary 1.

Concerned with the state-space realizations, Theorem 1 applies when no cancellation is present since the realizations are non minimal. The plant stability is automatically guaranteed from the strict positive realness- type inequalities invoked in Propositions 3-4. Corollary 1 extends the result of absolute stability to stabilizable and detectable realizations which cannot possess unstable zero/pole cancellations. Corollary 2 extends the absolute stability to all stable (cancellation-free or not) realizations since any stabilizable and detectable realization not being controllable and observable has always stable zero/pole cancellations. On the other hand, Proposition 3 required in Theorem 1 needs Assumption 1 as a necessary condition since if the delay-free nominal system is not controllable and observable then the state-space realization is non minimal. Similarly if Proposition 4 is invoked then Assumption $1'$ is required to guarantee the absence of unstable cancellations in the current delay-free system. However, both assumptions are not made explicit in Theorem 1 since they are ensured in terms of sufficiency-type constraints by the frequency-type inequalities of Proposition 3, or respectively Proposition 4. Assumptions 1 and $1'$ are relaxed to Assumptions 2 and $2'$ as corresponding implicit requirements in Corollary 1 since the nominal/current delay-free state-space realization is required to posses (at most) stable cancellations, if any, as a necessary condition for G(s) to have no unstable zero/pole cancellation. The various given Assumptions are useful as preliminary tests before checking the fulfillment's of Propositions 3-4. Theorem 1 and Corollaries 1-2 adopt simpler versions than those deriving from propositions 3-4 in the special case when c=Pb with being a positive definite symmetric real matrix. The subsequent result applies to the Principal Case for k = ∞ follows.

Theorem 2: Assume that c=Pb with $P = P^T > 0$ and the unforced nominal-delay free system is exponentially stable satisfying the Lyapunov matrix equality $A^T P + PA = -Q = -Q^T < 0$ with $|\delta| < \delta_0 = \left(\left\| (sI - A)^{-1} \left(\sum_{i=1}^{r} A_i e^{-h_i s} \right) \right\|_\infty \right)^{-1}$ and that the transfer function of the nominal delay-free system $G_0(s) = b^T P (sI - A)^{-1} b + d$ is strictly positive real (i.e. Re G_0 (s) > 0 for Re s $\geq$ 0 and G_0 (s) is strictly stable). Thus, G(s) is strictly positive real and any stabilizable and detectable state-space realization of the forward linear system (1. a) -(1.b) under nonlinear feedback (1.c) is absolutely stable for all nonlinear devices within the sector $[0, \infty)$. The result also applies to any realization which is not stabilizable-detectable possessing only stable zero/pole cancellations in its transfer function.

A similar result to Theorem 2 may be obtained for the Principal Case by assuming the strict positive realness of the transfer function of the current delay-free system from Proposition 4 for q=0. Since a Popov´s-type integral inequality:

$$\int_0^t y(\tau)\, \Phi(y(\tau))\, d\tau \geq -\gamma_0^2$$

is satisfied for all time and the transfer function of the time-invariant forward loop is strictly positive real then the system is, furthermore,asymptotically hyper stable independent of the delays (which is a stronger property than absolute stability[3,6] what means that Lyapunov´s global stability holds even if the nonlinear device (1.c) is time-varying. If the forward-loop transfer function is positive real, rather than strictly positive real, the closed-loop system associated with the Principal Case is hyperstable (i.e. globally Lyapunov´s stable for any nonlinear device (1.c) satisfying the above Popov´s-. type integral inequality).

EXAMPLES

Example 1: Consider (1) with n= 2, r = 1; $b^T = (-1\ ,\ \beta)\ k$; $c^T = (1,\ 0)$ and $A = \begin{bmatrix} 0 & 1 \\ -a\beta & -(\beta + a) \end{bmatrix}$; $A_1 = a_1 \begin{bmatrix} 0 & 1 \\ -a & -1 \end{bmatrix}$ with a > 0, $\beta > 0$ and d > 0. The open-loop forward-loop is globally asymptotically stable if $|\delta| < \left| \frac{a}{a_1} \right|$ and A is a stability matrix satisfying $A^T P + PA = -L$ for some real n-matrix $P = P^T > 0$ for any given real n-matrix $L = L^T > 0$. It is well-known that $P = \int_0^\infty e^{A^T\tau} L\, e^{A\tau} d\tau$. A simple calculation yields that Proposition 3 holds with $q = \frac{Pb - c}{\sqrt{d}}$ δ_0 is calculated by using $\alpha \leq \alpha_0 |a\ a_1|$ and $\alpha_0 \leq \sqrt{1 + \left(\frac{(\beta + a)^2 + (\beta + a)}{a^2 \beta^2} \right)}$ obtained from the calculations of the related H_∞- norms. Thus, from Theorem 1, the feedback system (1) is absolutely stable for any nonlinear device satisfying either in $[0, \infty)$ under feedback (1.c) or in $(0, \infty)$ for the feedback law (1.d). The associate transfer function posses a strictly stable zero/pole cancellation at s = - β which has not been taken into account in the above calculations. This is reasonable when the transfer function numerator and denominator are not factored explicitly from the state-space description especially for higher order systems. If such a cancellation is known and removed for a minimum state-space realization of (1.a)-(1.b) resulting in A=-a, $A_1 = -a$, b=k, c=1 then $\delta_0 = \frac{da^2}{|a_1| (da + 2k(a+1)a)}$ with P = 1/k,q=0. In this simple example, the calculations may also be performed from the real part of the transfer function once the cancellation, if known, is removed. In this case, this leads to d > 0, $\delta_0 = \frac{a}{|a_1|}$ wfoundis the weakest found constraint. However, obtaining factored transfer functions from a state-space realization is not direct for high-order systems in the presence of delays. This fact justifies the adequacy of the proposed method to practical problems.

Example 2: Assume that the transfer function of (1. a) -(1.b) is first-order and single-delayed given by $G(s) = \frac{cb}{s + a - \delta\, a_1 e^{-hs}} + d$. If d > 0 and c = pb for any real p>0 then the open-loop linear system is Lyapunov´s asymptotically stable independent of the size of the delay h if $a > 0$ and $|\delta| \in [0, \delta_0)$ with $\delta_0 \leq \frac{a}{|a_1|}$ from

Proposition 1. Furthermore, the system (1. a) -(1.c) is absolutely stable in [0, ∞) since Proposition 3 and also both Theorems 1-2 hold with q=0. Since the transfer function of the time-invariant forward loop is positive real then the closed-loop configuration (1. a) -(1.c) is, in addition, asymptotically hyperstable [3,6], i.e. The nonlinear function may be even time-varying while satisfying a Popov´s-type integral inequality $\int_0^t y(\tau)\ \Phi\left(y(\tau)\right)\, d\tau \geq -\gamma_0^2$ for all time and the closed-loop system is globally Lyapunov´s stable Assume that in this example $c \neq p\,b$ but $d > \left| \frac{c\,b}{a-\delta_0 \left| a_1 \right|} \right|$. Thus, asymptotic hyperstability and then absolute stability in [0, ∞) still hold.

Example 3: Assume that the transfer function of (1. a) -(1.b) is a second-order one in the Simplest Particular Case and single-delayed given by $G(s) = \frac{c\,b}{s\left(s+a_1-\delta\, e^{-hs}\right)} + \frac{d}{s}$. Thus, the closed-loop system (1. a) -(1, b), (1.d) is absolutely stable in (0, ∞) from Theorem 1, under Proposition 3, with q > 0 if $|\delta| \in \left[0, \delta_0\right)$ with $\delta_0 \leq \left|a_1\right| = \lambda a < a$ with a > 0 and $0 \leq \lambda < 1$ (a is an absolute upper-bound of a_1) provided that $d > \frac{(\lambda-1)a + q\lambda^2 a^2/4}{(1+\lambda)^2 a^2}$.

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