American Journal of Applied Sciences 9 (4): 584-592, 2012
ISSN 1546-9239

Some Applications of Spanning Trees in Complete and Complete Bipartite Graphs

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Abstract: Problem statement: The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant, it is also an important measure of reliability of a network. **Approach:** Using linear algebra and matrix analysis techniques to evaluate the associated determinants. **Results:** In this study we derive simple formulas for the number of spanning trees of complete graph K_n and complete bipartite graph $K_{n,m}$ and some of their applications. A large number of theorems of number of the spanning trees of known operations on complete graph K_n and complete bipartite graph $K_{n,m}$ are obtained. **Conclusion:** The evaluation of number of spanning trees is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the following important theorems and lemmas and their proofs.

Key words: Complete graph, complete bipartite graph, spanning trees, Kirchhoff matrix, operations on graphs

INTRODUCTION

We consider finite undirected graph with no loops or multiple edges. Let G be such a graph on n vertices. A spanning tree for a graph G is a subgraph of G that is a tree and contains all vertices of G. There are many situation in which good spanning trees must be found. Whenever one wants to find a simple, cheap, yet efficient way to connect a set of terminals, be they computers, telephones, factories, or cities, a solution is normally one kind of spanning trees. Spanning trees prove important for several reasons: They create a spare subgraph that reflects a lot about the original graph, they play an important role to designing efficient routing algorithms, some computationally hard problems, such as the Steiner tree problem and the travelling salesperson problem, can be solved by using spanning trees and they have wide applications in many areas such as network design, bioinformatics (Biggs, 1993; Brown *et al*., 1996; Colbourn, 1987; Bermond *et al*., 1995; Myrvold *et al*., 1991). The number of spanning trees of G, denoted by $\tau(G)$, is the total number of distinct spanning subgraphs of G that are trees. A classic result of Kirchhoff, (Cayley, 1889) can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = v_1,\ldots\ldots\ldots,v_n$. To state the result, we define the $n\times n$ characteristic matrix $A = [a_{ij}]$ as follows: (i) $a_{ij} = -1$ if v_i and v_j are adjacent and $i \neq j$, (ii) a_{ij} equals the degree of vertex v_i if $i = j$ and (iii) $a_{ij} = 0$ otherwise. The Kirchhoff matrix tree theorem states that all cofactors of A are equal and their common value is $\tau(G)$. The matrix tree theorem can be applied to any graph G to determine $\tau(G)$, but this requires evaluating a determinant of a corresponding characteristic matrix. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees especially when these numbers are very large. One of the first such result is due to Cayley who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees (Cvetkovic *et al*., 1980) that he showed $\tau(K_n) = n^{n-2}$, $n \geq 2$. Another result, $\tau(K_n) = p^{q-1}$, $q^{p-1}p,q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices,

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respectively. It is well known, as in e.g., (Petingi *et al*., 1998; Austin, 1960; Clark, 2003; Egecioglu and Remmel, 1994; Porter, 2004; Lewis, 1999).

Let G_1 and G_2 be a simple graphs. We introduce some operations on graphs:

- If G_1 and G_2 are vertex disjoint graphs. Then the join product, $G_1 \vee G_2$, is the super-graph of $G_1 + G_2$, in which each vertex of G_1 is adjacent to every vertex of G_2, (Balakrishnan and Ranganathan, 2000).
- The Cartesian product of two graphs G_1 and G_2, $G_{1\times} G_2$, is the simple graph with vertex set $V(G_{1\times} G_2) = V\ (V_1 \times V_2)$ and edge set $E(G_1 \times G_2) = [(E_1 \times V_2) \cup (V_1 \times E_1)]$, such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ iff, either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_1, or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, (Wilson and Watkins, 1990).
- The tensor product, or Kronecher product of two graphs G_1 and G_2, $G_1 \otimes G_2$, is the simple graph with $v\ (G_1 \otimes G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ iff u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2, (Balakrishnan and Ranganathan, 2000).
- The normal product, or the strong produc of two graphs G_1 and G_2 t, $G_1 \circ G_2$, is the simple graph with $V(G_1 \circ G_2) = V_1 \times V_2$ where (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \circ G_2$ iff either $u_1 = v_1$ and u_2 is adjacent to v_2 inG_2, or u_1 is adjacent to v_1 in G_1 and $u_2 = v_2$, or u_1 is adjacent to v_1 and u_2 is adjacent to v_2, (Balakrishnan and Ranganathan, 2000).
- The corona $G_1 \odot G_2$ of G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has n_1 vertices) and n_1 copies of G_2 and then joining the i^{th} vertex of G_2 to every vertex in the i^{th} copy of G_2, (Wilson and Watkins, 1990).

The well-known matrix tree theorem (Kirchhoff matrix) can be used to count the number of spanning trees for small graphs, but this method is not feasible for large graphs. So we present two formulas in lemma1, lemma2 is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem.

Lemma 1: Let G be a simple graph with n vertices. Then, $\tau(G) = \frac{1}{n^2}\det[D - A + U]$, where A, D are the adjacency and degree matrices of G respectively and U is the $n \times n$ matrix where all its elements are ones.

Proof: By simple calculations using the property of addition of two determinants, we can write det[D-A+U] as an addition of n determinants each of which is the same as det[D-A], but by replace one of its column and any determinant of these is equal to n× det[D-A]. Since all cofactors of [D-A] are equals. Then we have:

$$\det[D - A + U] = n \times \text{cofactor}[D - A] + n \times \text{cofactor}[D - A] + \ldots\ldots + n \times \text{cofactor}[D - A](n - \text{times}) + \det[D - A]$$

But:

$$\det[D - A] = 0$$

Then:

$$\det[D - A + U] = n^2 \times \text{cofactor}[D - A] = n^2 \tau(G)$$

Therefore:

$$\tau(G) = \frac{1}{n^2}\det[D - A + U]$$

Lemma 2: Let G be a simple graph with vertices n. Then, $\tau(G) = \frac{1}{n^2}\det[nI_n - \bar{D} + \bar{A}]$, where $\bar{A}, \bar{D}$ are the adjacency and degree matrices of $\bar{G}$ (complement of G) respectively and I_n is the identity matrix.

Proof: It is clear that $D + \bar{D} = (n-1)I_n$ and $(U - A - I_n) = \bar{A}$ .Then. $D - A + U = (n-1)I_n - \bar{D} + \bar{A} + I_n = nI_n - \bar{D} + \bar{A}$. Thus from lemma1, we have: $\tau(G) = \frac{1}{n^2}\det[nI_n - \bar{D} + \bar{A}]$.

Theorem 3: $\tau(K_n) = n^{n-2}$.

Proof: Applying lemma 2, we have:

$$\tau(K_n) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A}) = \frac{1}{n^2}\det\begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & n & 0 \\ 0 & \cdots & 0 & 0 & n \end{pmatrix} = \frac{1}{n^2} \times n^n = n^{n-2}$$

Corollary 4: $\tau(K_n - e) = (n-2)n^{n-3}$.

Proof: Applying lemma 2, we have:

$$\tau(K_n - e) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A})$$

$$= \frac{1}{n^2}\det\begin{pmatrix} n-1 & 1 & 0 & \cdots & 0 \\ 1 & n-1 & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & n & 0 \\ 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

$$= \frac{1}{n^2}\times((n-1)^2 - 1)\times n^{n-2} = (n-2)\times n^{n-3}$$

Corollary 5: $\tau(K_n \circ e) = 2n^{n-3}$, where $K_n \circ e$ is the graph obtained from K_n by contracting the edge e.

Proof: Immediately from the fact that:

$$\tau(G) = \tau(G - e) + \tau(G \circ e)$$

Theorem 6: Let G be a graph constructed by removing m distinct edges from $K_n, n \geq 2m$. Then:

$$\tau(G) = n^{n-2}(1 - \frac{2}{n})^m$$

Proof: Straightforward induction using properties of determinants.

Theorem 7: $\tau(K_n + e) = (n+2)n^{n-3}$.

Proof: Applying lemma 2, we have:

$$\tau(K_n + e) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A}) =$$

$$\frac{1}{n^2}\det\begin{pmatrix} n+1 & -1 & 0 & \cdots & 0 \\ -1 & n+1 & 0 & \ddots & \vdots \\ 0 & \ddots & n & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & n \end{pmatrix}$$

$$= \frac{1}{n^2}\times((n+1)^2 - 1)\times n^{n-2} = (n+2)\times n^{n-3}$$

Theorem 8: Let H_n be a graph constructed by removing n distinct edges from K_{2n}. Then:

$$\tau(H_n) = 2^{2n-2}\times n^{n-1}\times(n-1)^n$$

Proof: Applying lemma 2, we have:

$$\tau(H_n) = \frac{1}{(2n)^2}\det(2nI - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$

$$\begin{pmatrix} 2n-1 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 2n-1 & 0 & \ddots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 0 & 2n-1 & 1 & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 2n-1 & 0 & \ddots & & & \ddots & \vdots \\ \vdots & \cdots & 0 & 0 & 2n-1 & 1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & & \ddots & 1 & 2n-1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & & \ddots & 0 & \ddots & 0 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \vdots & 0 & 1 & \ddots & & 0 \\ & & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & 2n-1 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 2n-1 \end{pmatrix}$$

Straightforward induction using properties of determinants. We have:

$$\tau(H_n) = \frac{1}{(2n)^2}\times((2n-1)^2 - 1)^n = 2^{2n-2}\times n^{n-1}\times(n-1)^n$$

Theorem 9: Let G be a graph constructed by removing a star graph $K_{1,2}$ from K_n. Then:

$$\tau(G) = n^{n-2}(1 - \frac{1}{n})(1 - \frac{3}{n})$$

Proof: Apply lemma 2, we have:

$$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A}) = \frac{1}{n^2}\det$$

$$\begin{pmatrix} n-2 & 1 & 1 & 0 & \cdots & \cdots & \cdots \\ 1 & n-1 & 0 & \ddots & \vdots & \ddots & \ddots \\ 1 & 0 & n-1 & 0 & 0 & \vdots & \ddots \\ 0 & \ddots & 0 & n & 0 & \ddots & \\ \vdots & \cdots & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & & \ddots & 0 & n \end{pmatrix}$$

$$= \frac{1}{n^2}\det\begin{pmatrix} n-2 & 1 & 1 \\ 1 & n-1 & 0 \\ 1 & 0 & n-1 \end{pmatrix}\times\det\begin{pmatrix} n & 0 & \cdots \\ 0 & n & \ddots \\ \vdots & \ddots & n \end{pmatrix}$$

$$= \frac{1}{n^2}\times n^{n-3}\times(n-1)(n^2 - 3n) = n^{n-2}\times(1 - \frac{1}{n})\times(1 - \frac{3}{n})$$

Corollary 10: Let G be a graph constructed by removing a star graph $K_{l,m}$ from Kn, n≥m+1.. Then:

$$\tau(G) = n^{n-2}(1 - \frac{1}{n})^{m-1}(1 - \frac{m+1}{n})$$

Proof: Straightforward induction using properties of determinants.

Lemma 11: Biggs (1993) let G be a k-regular graph with n vertices and m edges. Then:

$$\tau(L(G)) = 2^{m-n+1} \times k^{m-n-1} \times \tau(G)$$

where, $L(G)$ is the line graph of G.

Theorem 12: Let T_n be the line graph of K_n. Then:

$$\tau(T_n) = \tau(L(K_n)) = 2^{\frac{1}{2}(n^2-3n+2)} \times (n-1)^{\frac{1}{2}(n^2-3n-2)} \times n^{n-2}$$

Proof: The line graph $L(G)$ of a graph G is constructed by taking the edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$ whenever the corresponding edges in G have a common vertex. Also if G is regular of valency k, its line graph $L(G)$ is regular of valency $2k-2$. It is easy to show that $L(K_n)$ is the triangle graph T_n which can be described by saying that the $\frac{1}{2}n(n-1)$ pairs of numbers from the set $\{1, 2, ..., n\}$, two vertices being adjacent whenever the corresponding pairs have just one common member. Appling lemma11 taking $k = n-1$, $m = \frac{1}{2}n(n-1)$, we have:

$$\tau(T_n) = \tau(L(K_n)) = 2^{\frac{1}{2}(n^2-3n+2)} \times (n-1)^{\frac{1}{2}(n^2-3n-2)} \times n^{n-2}$$

Lemma 13: Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} x & 1 & 1 & \cdots & \cdots & 1 \\ 1 & x & 1 & \ddots & & \vdots \\ 1 & \ddots & \ddots & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & & \ddots & \ddots & x & 1 \\ 1 & \cdots & \cdots & 1 & 1 & x \end{pmatrix}$$

Then:

$$\det(A_n) = (x+n-1)(x-1)^{n-1}$$

Proof: From the definition of the circulant determinants, we have:

$$\det(A_n(x)) = \prod_{j=1}^{n} (x + \omega_j + \omega_j^2 + \omega_j^3 + ... + \omega_j^{n-1})$$
$$= (x+1+1+.......+1) \times \prod_{j=1, \omega_j \neq 1}^{n} (x + \underbrace{\omega_j + \omega_j^2 + \omega_j^3 + ... + \omega_j^{n-1}}_{=-1})$$
$$= (x+n-1) \times (x-1)^{n-1}.$$

Lemma14: If $H = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ and AB = BA. Then det(H) = det(A+B). det (A-B)**.**

Proof: Using the fact that $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{cases} \det(A).\det(D - CA^{-1}B) \\ \det(D).\det(A - BD^{-1}C) \end{cases}$**,** where A,B are non singular, Marcus M. [12]. We have

$$\det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det(A).\det(A - BA^{-1}B)$$
$$= \det(A^2 - B^2) = \det(A+B).\det(A-B).$$

This formula gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

Theorem 15: $\tau(K_2 \times K_n) = n^{n-2} \times (n+2)^{n-1}$.

Proof: Applying lemmas 2. We have:

$$\tau(K_2 \times K_n) = \frac{1}{(2n)^2}\det(2nI - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$

$$\begin{pmatrix} n+1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & n+1 & 0 & \ddots & \vdots & 1 & 0 & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & n+1 & 0 & \vdots & & 1 & 0 & 1 \\ 0 & \cdots & \cdots & 0 & n+1 & 1 & \cdots & \cdots & 1 & 0 \\ 0 & 1 & \cdots & \cdots & 1 & n+1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 1 & \cdots & \vdots & 0 & n+1 & 0 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & \ddots & & \vdots \\ & & \ddots & 0 & 1 & \vdots & \ddots & \ddots & n+1 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 & n+1 \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2 \times K_n) = \frac{1}{(2n)^2}\det\begin{pmatrix} n+1 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{pmatrix} \times \det\begin{pmatrix} n+1 & -1 & \cdots & \cdots & -1 \\ -1 & n+1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & n+1 & -1 \\ -1 & \cdots & \cdots & -1 & n+1 \end{pmatrix}$$

Applying lemma13 with $x = n+1$, for the first determinant and properties of determinants for the second, yields:

$$\tau(K_2 \times K_n) = \frac{1}{(2n)^2} \times 2n^n \times 2(n+2)^{n-1} = (n+2)^{n-1} \times n^{n-2}$$

Theorem 16: $\tau(K_2 \otimes K_n) = n^{n-2}(n-1)(n-2)^{n-1}$.

Proof: Applying lemmas 2. We have:

$$\tau(K_2 \otimes K_n) = \frac{1}{(2n)^2}\det(2n\,I - \bar{D} + \bar{A}) =$$

$$\frac{1}{(2n)^2}\det\begin{pmatrix} n & 1 & \cdots & 1 & & & & \\ 1 & \ddots & & \vdots & & & I_n & \\ & & \ddots & 1 & & & & \\ 1 & \cdots & 1 & n & & & & \\ & & & & n & 1 & \cdots & 1 \\ & & & & 1 & \ddots & & \vdots \\ & I_n & & & \vdots & & \ddots & 1 \\ & & & & 1 & \cdots & 1 & n \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2 \otimes K_n) =$$
$$\frac{1}{(2n)^2}\det\begin{pmatrix} n+1 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{pmatrix} \times \det\begin{pmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 1 & n-1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n-1 & 1 \\ 1 & \cdots & 1 & 1 & n-1 \end{pmatrix}$$

Applying lemma13 with $x = n+1, n-1$ respectively, we have:

$$\tau(K_2 \otimes K_n) = \frac{1}{(2n)^2} \times 2n^n \times 2(n-1)(n-2)^{n-1}$$
$$= n^{n-2}(n-1)(n-2)^{n-1}$$

Theorem 17: $\tau(K_2 \circ K_n) = (2n)^{2n-2}$.

Proof: Applying lemma 2, we have:

$$\tau(K_2 \circ K_n) = \frac{1}{(2n)^2}\det(2n\,I - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$
$$\begin{pmatrix} 2n & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 2n & 0 & \ddots & \vdots & \vdots & 0 & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2n & 0 & \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 2n & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 2n & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \cdots & \cdots & \vdots & 0 & 2n & 0 & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & \ddots & & \vdots \\ & & \ddots & \cdots & \vdots & \vdots & \ddots & \ddots & 2n & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 2n \end{pmatrix}$$
$$= \frac{1}{(2n)^2}(2n)^n \times (2n)^n = (2n)^{2n-2} = \tau(K_{2n})$$

Theorem 18: $\tau(K_2 \odot K_n) = n^{2(n-2)}$.

Proof: Applying lemma 2, we have:

$$\tau(K_2 \odot K_n) = \frac{1}{(2n)^2}\det(2n\,I - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$

$$\begin{pmatrix} n+1 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & n & 0 & \ddots & \vdots & 1 & 1 & \cdots & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & n & 0 & \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & n & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 & n+1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & \vdots & 0 & n & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \cdots & \vdots & \vdots & \ddots & \ddots & n & 0 \\ 1 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & n \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2 \odot K_n) =$$
$$\frac{1}{(2n)^2}\det\begin{pmatrix} n+1 & 1 & 1 & 1 \\ 1 & n+1 & 1 & \vdots \\ 1 & 1 & \ddots & \vdots \\ \vdots & \cdots & \ddots & n+1 \end{pmatrix} \times \det\begin{pmatrix} n+1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \ddots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{pmatrix}$$

Applying lemma13 with $x = n+1$ for the first determinant and properties of determinants for the second, yields:

$$\tau(K_2 \odot K_n) = \frac{1}{(2n)^2} \times 2n^n \times 2n^{n-2} = n^{2(n-2)}$$

Theorem 19: $\tau(K_{n,m}) = n^{m-1}m^{n-1}$.

Proof: Apply lemma 2, we have:

$$\tau(K_{n,m}) = \frac{1}{(n+m)^2}\det((n+m)I - \bar{D} + \bar{A}) = \frac{1}{(n+m)^2}\det$$
$$\begin{pmatrix} m+1 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \vdots \\ 1 & \ddots & 1 & m+1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & n+1 & 1 & \cdots & 1 \\ \vdots & \cdots & \cdots & \vdots & 1 & \ddots & \vdots & \vdots \\ \vdots & \cdots & \cdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+1 \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_{n,m}) =$$
$$\frac{1}{(n+m)^2}\det\begin{pmatrix} m+1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & m+1 \end{pmatrix} \times \det\begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & n+1 \end{pmatrix}$$

Applying lemma13 with $x = n + 1$ and $x = m + 1$ respectively, we have:

$$\tau(K_{n,m}) = \frac{1}{(n+m)^2}\cdot$$
$$(n+m)\times m^{n-1}\times(n+m)\times n^{m-1} = m^{n-1}n^{m-1}$$

Specially, $\tau(K_{n,n}) = n^{2n-2}$.

Corollary 20: $\tau(K_{n,n} - e) = n^{2n-4}\times(n-1)^2$.

Proof: Applying lemma 2, we have:

$$\tau(K_{n,n} - e) = \frac{1}{(2n)^2}\det(2n\,I - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$
$$\begin{pmatrix} n & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \vdots & n+1 & \vdots & \vdots & 0 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \vdots \\ 1 & \ddots & 1 & n+1 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & 0 & n & 1 & \cdots & 1 \\ 0 & 0 & \cdots & \vdots & 1 & n+1 & \vdots & \vdots \\ \vdots & \cdots & \cdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+1 \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_{n,n} - e) =$$
$$\frac{1}{(2n)^2}\det\begin{pmatrix} n+1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & n+1 \end{pmatrix}\times\det\begin{pmatrix} n-1 & 1 & \cdots & 1 \\ 1 & n+1 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & n+1 \end{pmatrix}$$

Applying lemma13 with $x = n+1$ for the first determinant and properties of determinants for the second, yields:

$$\tau(K_{n,n} - e) = \frac{1}{(2n)^2}\times 2n^n\times 2n^{n-2}\times(n-1)^2 = n^{2n-4}\times(n-1)^2$$

Theorem 21: $\tau(K_{n,n} + e) = n^{2(n-2)}(n^2 + 2n - 1)$.

Proof: Applying lemma 2, we have:

$$\tau(K_{n,n} + e) = \frac{1}{(2n)^2}\det(2n\,I - \bar{D} + \bar{A}) = \frac{1}{(2n)^2}\det$$
$$\begin{pmatrix} n+2 & 1 & \cdots & \cdots & 1 & -1 & 0 & \cdots & \cdots & 0 \\ 1 & n+1 & 1 & \ddots & \vdots & 0 & 0 & \cdots & \cdots & \vdots \\ \vdots & 1 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & n+1 & 1 & \vdots & \cdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & n+1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & \cdots & \cdots & 0 & n+2 & 1 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & \vdots & 1 & n+1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \cdots & \vdots & \vdots & \ddots & \ddots & n+1 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & n+1 \end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_{n,n} + e) =$$
$$\frac{1}{(2n)^2}\det\begin{pmatrix} n+1 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{pmatrix}\times\det\begin{pmatrix} n+3 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{pmatrix}$$

Applying lemma13 with $x = n+1$ for the first determinant and properties of determinants for the for the second, yields:

$$\tau(K_{n,n} + e) = \frac{1}{(2n)^2}\times 2n^n\times 2n^{n-2}(n^2 + 2n - 1)$$
$$= n^{2(n-2)}(n^2 + 2n - 1)$$

Theorem 22: $\tau(L(K_{n,n})) = 2^{n^2-2n+1}\times n^{n^2-2n-1}\times n^{2(n-1)}$.

Proof: It is easy to show that $L(K_{n,n})$ is the $K_n\times K_n$.
Appling lemma11 taking $k = n$ and $m = n^2$.
We have:

$$\tau(L(K_{n,n})) = \tau(K_n\times K_n) = 2^{n^2-2n+1}\times n^{n^2-2n-1}\times n^{2(n-1)}$$

Theorem23:

$$\tau(K_1 + K_{n,m}) = (n+m+1)\times(n+1)^{m-1}\times(m+1)^{n-1}.$$

Proof: Applying lemma 2, we have:

$$\tau(K_1 + K_{n,m}) = \frac{1}{(n+m+1)^2}\det[(n+m+1)I - \bar{D} + \bar{A}]$$
$$= (n+m+1)\times\frac{1}{(n+m+1)^2}\det$$
$$\begin{pmatrix} m+2 & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \cdots & \cdots & \vdots & \vdots \\ 1 & \ddots & 1 & m+2 & 0 & \cdots & \cdots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & n+2 & 1 & \cdots & 1 & \vdots \\ \vdots & \cdots & \cdots & \vdots & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \cdots & \cdots & \vdots & \vdots & \ddots & \ddots & 1 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & 1 & n+2 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & n+m+1 \end{pmatrix}$$

$$= (n+m+1)\times\frac{1}{(n+m+1)^2}\det$$

$$\begin{pmatrix} m+2 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & m+2 \end{pmatrix}\times\det\begin{pmatrix} n+2 & 1 & \cdots & 1 \\ 1 & n+2 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & n+2 \end{pmatrix}$$

Applying lemma 13 with $x = m + 2$ and $x = n + 2$, respectively, we have:

$$\tau(K_1 + K_{n,m}) = (n+m+1)\times(n+1)^{m-1}\times(m+1)^{n-1}$$

Specially, $\tau(K_1 + K_{n,n}) = (2n+1)\times(n+1)^{2n-2}$.

Corollary24: $\tau(K_{n,n}\circ e) = (2n-1)n^{2n-4} = \tau(K_1 + K_{n-1,n-1})$.

Theorem 25: $\tau(K_2 \times K_{n,n}) = 2n^{2n-2}\times(n+1)\times(n+2)^{2n-2}$.

Proof: Applying lemma 2, we have:

$$\tau(K_2\times K_{n,n}) = \frac{1}{(4n)^2}\det(4n\,I - \bar{D} + \bar{A}) = \frac{1}{(4n)^2}\det$$

$$\begin{pmatrix}
n+2 & 0 & 1 & 0 & 1 & \cdots & \cdots & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
0 & n+2 & 0 & 1 & 0 & \ddots & \ddots & 1 & 0 & 1 & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & n+2 & 0 & 1 & 0 & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 0 & \ddots & 0 & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & \ddots & \ddots & \ddots & 1 & 1 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & 1 & 0 & n+2 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 1 \\
\vdots & \cdots & \cdots & \vdots & 1 & 0 & n+2 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & n+2 & 0 & 1 & 0 & 1 & \cdots & \cdots \\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & n+2 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 1 & 0 & n+2 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & n+2 & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & n+2
\end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2\times K_{n,n}) = \frac{1}{(4n)^2}\det\begin{pmatrix}
n+2 & 1 & 2 & 1 & 2 & \cdots & \cdots \\
1 & n+2 & 1 & \ddots & \ddots & \ddots & \ddots \\
2 & 1 & n+2 & \ddots & 2 & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & n+2 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & n+2
\end{pmatrix}\times$$

$$\det\begin{pmatrix}
n+2 & -1 & 0 & -1 & \cdots & \cdots & \cdots \\
-1 & n+2 & -1 & \ddots & \ddots & \ddots & \vdots \\
0 & -1 & n+2 & \ddots & \ddots & \ddots & \vdots \\
-1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 & n+2 & -1 \\
\vdots & \cdots & \cdots & \cdots & 0 & -1 & n+2
\end{pmatrix}$$

Straightforard induction using properties of determinants.

We have:

$$\tau(K_2\times K_{n,n}) = \frac{1}{(4n)^2}\times 8n^{2n}\times 4(n+1)(n+2)^{2n-2}$$
$$= 2n^{2n-2}\times(n+1)\times(n+2)^{2n-2}$$

Theorem 26: $\tau(K_2\circ K_{n,n}) = 2^{4n-2}\times n^{2n-2}\times(n+1)^{2n}$.

Proof: Apply lemma 2, we have:

$$\tau(K_2\circ K_{n,n}) = \frac{1}{(4n)^2}\det(4n\,I - \bar{D} + \bar{A}) = \frac{1}{(4n)^2}\det$$

$$\begin{pmatrix}
2n+2 & 0 & 1 & 0 & 1 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 & \cdots & \cdots \\
0 & 2n+2 & 0 & 1 & 0 & \ddots & \ddots & 0 & 0 & 0 & 1 & 0 & 1 & \ddots \\
1 & 0 & 2n+2 & 0 & 1 & 0 & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & 0 & \ddots & 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & \cdots & \ddots & \ddots & \ddots & 1 & 1 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \cdots & \cdots & 1 & 0 & 2n+2 & 0 & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \cdots & \cdots & \vdots & 1 & 0 & 2n+2 & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 1 & 0 & 1 & \cdots & \cdots & 2n+2 & 0 & 1 & 0 & 1 & \cdots & \cdots \\
0 & 0 & 0 & 1 & 0 & \ddots & \cdots & 0 & 2n+2 & 0 & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & 0 & 0 & \ddots & \ddots & \ddots & 1 & 0 & 2n+2 & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 2n+2 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 2n+2
\end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2\circ K_{n,n}) = \frac{1}{(4n)^2}\det\begin{pmatrix}
2n+2 & 0 & 2 & 0 & 2 & \cdots & \cdots \\
0 & 2n+2 & 0 & \ddots & \ddots & \ddots & \ddots \\
2 & 0 & 2n+2 & \ddots & 2 & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2n+2 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 2n+2
\end{pmatrix}\times$$

$$\det\begin{pmatrix}
2n+2 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 2n+2 & 0 & \ddots & \ddots & & \vdots \\
\vdots & 0 & 2n+2 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 2n+2 & 0 \\
0 & \cdots & \cdots & \cdots & \ddots & 0 & 2n+2
\end{pmatrix}$$

Straightforward induction using properties of determinants.

We have:

$$\tau(K_2 \circ K_{n,n}) = 2^{4n-2} \times n^{2n-2} \times (n+1)^{2n}$$

Theorem 27: $\tau(K_2 \odot K_{n,n}) = (n+1)^{4n-4} \times (2n+1)^2$.

Proof: Applying lemma 2, we have:

$$\tau(K_2 \odot K_{n,n}) = \frac{1}{4(2n+1)^2}\det((4n+2)I - \bar{D} + \bar{A}) = \frac{1}{4(2n+1)^2}\det$$

$$\begin{pmatrix}
2n+2 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
0 & n+2 & 0 & 1 & 0 & \ddots & \vdots & 1 & 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & n+2 & 0 & 1 & \ddots & \vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & 0 & \ddots & 0 & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & n+2 & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \vdots & \ddots & \ddots & 1 & 0 & n+2 & 1 & \cdots & \cdots & \cdots & \cdots & \ddots & 1 \\
0 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 2n+2 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
1 & 1 & \cdots & \ddots & \ddots & \ddots & \vdots & 0 & n+2 & 0 & 1 & 0 & \cdots & \cdots \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & n+2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & n+2 & 0 \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & 0 & \vdots & \ddots & \ddots & 1 & 0 & n+2
\end{pmatrix}$$

Using lemma14, we get:

$$\tau(K_2 \odot K_{n,n}) = \frac{1}{4(2n+1)^2}\det\begin{pmatrix}
2n+2 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
1 & n+3 & 1 & 2 & 1 & \cdots & \cdots \\
1 & 1 & n+3 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & n+3 & 1 \\
1 & \vdots & \ddots & \ddots & \ddots & 1 & n+3
\end{pmatrix} \times$$

$$\det\begin{pmatrix}
2n+2 & -1 & -1 & \cdots & \cdots & \cdots & -1 \\
-1 & n+1 & -1 & 0 & -1 & \cdots & \cdots \\
-1 & -1 & n+1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & n+1 & -1 \\
-1 & \vdots & \cdots & \cdots & 0 & -1 & n+1
\end{pmatrix}$$

Straightforward induction using properties of determinants.

We have:

$$\tau(K_2 \odot K_{n,n}) = \frac{1}{4(2n+1)^2} \times 2(n+1)^{2n-2} \times (2n+1)^3$$
$$\times 2(n+1)^{2n-2} \times (2n+1) = (n+1)^{4n-4} \times (2n+1)^2$$

ACKNOWLEDGMENT

The researcher is deeply indebted to the team of work at deanship of scientific research at Taibah university, Al-Madinah Al-Munawwarah, K.S.A for their continuous helps and the encouragement us to finalize this study.

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