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Equivalent Torsional-Warping Stiffness of Cores with Thin-Walled Open Cross-Section Using the Vlasov Torsion Theory

Triantafyllos Konstantinos Makarios

Department of Civil Engineering, Aristotle University of Thessaloniki, Thessaloniki, Greece

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Email: makariostr@civil.auth.gr

Abstract: In order to calculate the equivalent torsional-warping stiffness of the Reinforced Concrete (RC) cores that have thin-walled open cross-section, a new analytical methodology, which combines the Vlasov torsion theory with the Bernoulli bending theory, is presented herein. As the basis of the calculations, we use the principal elastic reference system of the core from this we consider that is known. Furthermore, we consider that the principal start point of the open cross-section, the core's exact sectorial coordinates, as well as, the warping moment of inertia of the core are all known, also. Moreover, the two above-mentioned theories (Vlasov and Bernoulli) are together combining and in the end, the equivalent torsional-warping stiffness of the core has resulted. This torsional-warping stiffness of the core is very useful in the right simulation of a building that consists of frames, walls, and cores. The present methodology is presented via two special numerical cases of RC cores for illustrative purposes. The present article gives a documented solution in the simulation of the cores and proposes to use an ideal-equivalent column that has to be located on the elastic center of the core. This equivalent column must be provided with a diagonal, lateral-stiffness matrix that represents the properties of the real core and thus this lateral-stiffness matrix of the core is proposed. Finally, in order to check the reliability of the results of various analysis software, the proposed procedure can be used as a benchmark analysis method of cores.

Keywords: Cores, Warping Stiffness, Principal Elastic Reference System, Start Point of Thin-Walled Open Cross-Section, Vlasov Torsion Theory, Sectorial Coordinates, Warping Moment of Inertia, Bi-moment Normal Stresses

Introduction

The core is a structural element with a thin-walled open cross-section that starts from the base of multi-story Reinforced Concrete (RC) buildings and reaches the top of them. Furthermore, cores are one of the most common structural members that are used in multi-story buildings and especially are used very often either in low buildings, either tall buildings (at the lift location and staircase position) such as skyscrapers, towers, or special chimneys. A core is consisting of non-same-planed surface disks, which are connected at their edges. Hence, arise a prismatic surface structure or structure member, in which each-one disk operates in a mixed way, since, on the one hand, be loaded into its plane with seismic or wind lateral loadings, and on the other hand, also, it is loaded with significant torsional moments about the vertical axis, which come from the floor rotational vibrations about the vertical axis of the building. In addition, at floor levels, cores are connected with the floor diaphragms, which ensures the same rotation angle about the vertical axis (into each diaphragm). The cross-section of the core possesses its local elastic center (or local shear center) that does not coincide with the geometric center of the thin-walled cross-sections, but this is located in another position, which is far away from its geometric center. As a result of this peculiarity, cores have strong three-dimensional (spatial) behavior and significantly affect the torsional-translational behavior of the building that is loading with wind loadings or seismic excitations. The local elastic center of a core (as it is a structural member) significantly affects the location of the real or fictitious elastic center of the asymmetric multi-story buildings (Terzi and Athanatopoulou, 2021; 2023). It is well-known that the Saint-Venant Torsion is an absolutely different phenomenon from the Vlasov torsion theory (Vlasov, 2020). Indeed, the first torsion is a pure torsion that causes shear stresses on the open cross-section, only. This phenomenon is studied by analysis of a finite element model using six degrees of freedom per joint. On the contrary, the second torsion is a torsion-warping phenomenon that causes normal stresses on the same open cross-section thanks to

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bi-moments. Vlasov has confronted this phenomenon by inserting into calculations a more, the seventh, degree of freedom (something that does not exist in the classic software of the finite element method), where this degree of freedom represents the change (in elevation) of the rotation angle of the cross-section around the vertical axis (Vlasov, 2020). Moreover, for this reason, it is well-known that in such structures, the finite element method gives approximate results, because the torsion-warping phenomenon is ignored by this method.

The above-mentioned points have preoccupied the international scientific community in the past (and in the present). In the recent work (Makarios and Athanatopoulou, 2022) there is rich international literature about this matter, but the issue of the core simulation remains almost unresolved. In order to simulate the cores of the buildings (in the right way), the present article proposes the idea that must define an ideal-equivalent (to the core) column, that will be located on the local elastic center of the core. Next, we provide this equivalent column with the equivalent torsion stiffness, considering the torsion-warping resistance of the core. It is worth noting that the right evaluation of the core torsion stiffness affects the calculation of the response/behavior of the total building. As the suitable key of the present procedure, we use the recent technique, that permits the exact calculation (Makarios and Athanatopoulou, 2022) of the principal elastic reference system of the core, the principal Start Point of the cross-section, the exact values of the sectorial coordinates of the open cross-section of the examined core, as well as, the warping moment of inertia.

Materials

Reinforced Concrete (RC) or steel or aluminum or each other material that can be considered as homogenous and isotropic material is using for cores.

Methodology

A new exact technique for the calculation of the following properties has been published in another work (Makarios and Athanatopoulou, 2022):

a) Of local principal elastic reference system $K(I, II, III)$ of a core
b) of the principal start point $M_o(x_o, y_o)$ of the cross-section
c) Of diagrams of the coordinate functions $\xi(s)$ and $\eta(s)$ of the thin-walled open section relative to the gravity reference system $G\xi\eta z$
d) Of the diagram of the exact sectorial coordinates $\omega(s)$ with respect to the pole K (that is the elastic center of the cross-section) and based on the principal start point M_o of the thin-walled open cross-section, and
e) of the warping moment of inertia I_ω

For this reason, we consider that all these above-mentioned properties of the core are known. However, we worth noting that, in order to calculate the above-mentioned properties, the following steps must be applied:

i) Calculation of the location of the center of gravity, G, and the orientation of the principal axes ξ and η of the thin-walled open cross-section
ii) Calculation of the principal moments of inertia I_ξ and I_η of the thin-walled open cross-section about the principal axes ξ and η passing through the gravity center G of the cross-section of the core
iii) Calculation of diagrams of coordinate functions $\xi(s)$ and $\eta(s)$ of the thin-walled open section relative to the gravity reference system $G\xi\eta z$
iv) Calculation of the location of the local elastic center K (which is the local stiffness center) of the thin-walled open section, using the repetitive mathematical procedure that has been proposed at work (Makarios and Athanatopoulou, 2022)
v) Calculation of the location of the principal start point $M_o(x_o, y_o)$ of the thin-walled open section as well as of the sectorial coordinates $\omega(s)$ with respect to the pole K and based on the principal start point M_o of the thin-walled open cross-section
vi) Calculation of the numerical value of the warping moment of inertia (or warping constant) I_ω, of the thin-walled open section, according to Vlasov torsion theory

In the present article, the following steps are proposed to evaluate the equivalent torsional-warping stiffness of the core:

(1) An enforced rotation angle θ_{III} around the elastic center K is applied at all cores. The Equations are written at two end-legs of the core, always, and the final equivalent torsional-warping stiffness $k_{\theta,III}$ of the core is the mean value of the torsional-warping stiffnesses of the two end-legs of the core
(2) The horizontal displacement on the top of each examined leg, along the principal elastic I or II-axis, is formulated
(3) The shear force on the top of the examined leg is formulated
(4) The flexural moment at the base of the examined leg due to the above-mentioned shear force is formulated
(5) The normal stresses $\sigma_z(0,i)$ on the cross-section at the base of the core are formulated according to both theories, the Vlasov torsion theory and the Bernoulli bending theory

(6) Combining the above-mentioned equations, the torsional-warping stiffness $k_{\theta,III}$ of the core is produced at first approximation. Afterward, a re-calculation (second approximation) on the other end-leg of the core is needed. From these two approximative calculations, the mean value $k_{\theta,III}$ is produced

Torsional-Warping Stiffness of Core with Thin-Walled with Open Cross-Section Shaped ⊏. First Case

We consider the core of Fig. 1, which is a structural member of a single-story *RC* building that has a height equal to four meters (H = 4.00 m). The core possesses a symmetry axis, the *X*-axis. Noting that this is a core of a staircase and, also, is fixed at its foundation, into the ground.

The location of the gravity center *G*, is shown on the symmetry *X*-axis (Fig. 1). Hence, the *X*-axis constitutes simultaneously and the first principal direction of the cross-section, that is symbolized as ξ-*axis*. The other, second principal direction is perpendicular to the first and is symbolized as $\eta - axis$, while both principal axes have a common origin the gravity center *G*. The principal moments of inertia are I_ξ and I_η of the thin-walled open cross-section about the principal directions ξ η and passing through the gravity center *G* of the cross-section, while the product moment of inertia $I_{\xi\eta}$ of the cross-section is zero. Therefore, after the calculations (where a simple way of calculation proposed by Makarios and Athanatopoulou (2022), the principal moments of inertia I_ξ I_η have the following values:

$$I_\xi = 15.40159\,m^4, I_\eta = 6.12454\,m^4 \sqrt{a^2 + b^2}$$

Also, the area of the cross-section is A = 3.72 m^2. Next, in Fig. 2 the diagrams of coordinate functions $\xi(s)$ $\eta(s)$ and, relative to the principal gravity Cartesian reference system $G\xi\eta z$. Furthermore, the steps (*i*-*vi*) of the above-mentioned methodology have been applied, and after two repeats the final position of the local elastic center *K* has been calculated at a distance 1.60 m left of the core back, while has distance from the gravity center equal with 2.794 m, on the symmetry *X*-axis (Fig. 3). At the same Figure, we can see the location of the principal start point $M_o(x_o, y_o)$ of the thin-walled open section as well as of the exact sectorial coordinates $\omega(s)$ with respect to the pole *K* and based on the principal start point M_o of the thin-walled open cross-section. The *I*-axis, which has the origin of the elastic center *K*, is the principal elastic axis (that it coincides with the symmetry axis of the core), and the II-axis, which also has an origin of the elastic center *K*, is the second principal elastic axis (that it is parallel with the η-axis). The third principal elastic axis is vertical and through from the elastic center *K*, too. Hence, the three principal elastic axes *I*, *II*, and *III* create the local principal elastic reference system *K*(*I*, *II*, *III*) of the core. Finally, the numerical value of the warping moment of inertia I_ω has resulted (Makarios and Athanatopoulou, 2022):

$$I_\omega = 23.7496\,m^6$$

All the above-mentioned calculations (Fig. 1-3) are the necessary spadework that is based on the recent article (Makarios and Athanatopoulou, 2022). Afterward, from this point and below, we are following the steps according to the new present methodology.

In order to calculate the torsional-warping stiffness $k_{\theta,III}$ of the core, we consider that it is loaded at its top with an external torsional moment M_t, around the vertical principal elastic *III*-axis. Due to the torsional moment M_t, the thin-walled open cross-section of the core is rotated around its elastic center *K* per angle θ_{III}. In Fig. 4 we can see the rotation angle θ_{III} of the core and the displacement diagrams due to the external torsional moment M_t around III-axis of the Cartesian principal elastic reference system *K*(*I, II, III*), using the kinematic conditions of the cross-section (that behaves as diaphragm according to Vlasov Torsion Theory). Also, the bi-moment diagram B_K along the height of the core is given in Fig. 5.

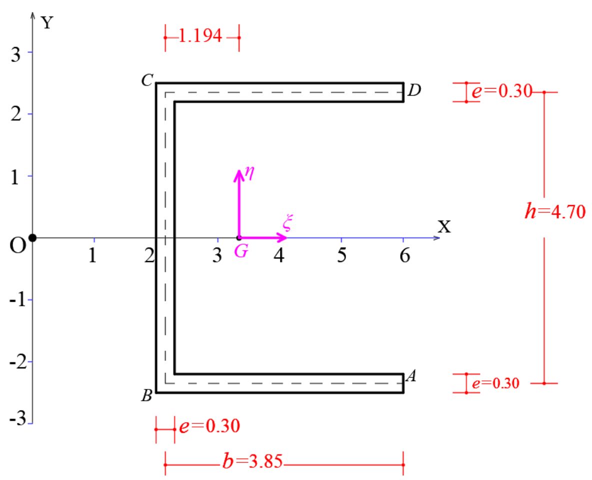

Fig. 1: Geometry of the core with shape ⊏ (units in meters)

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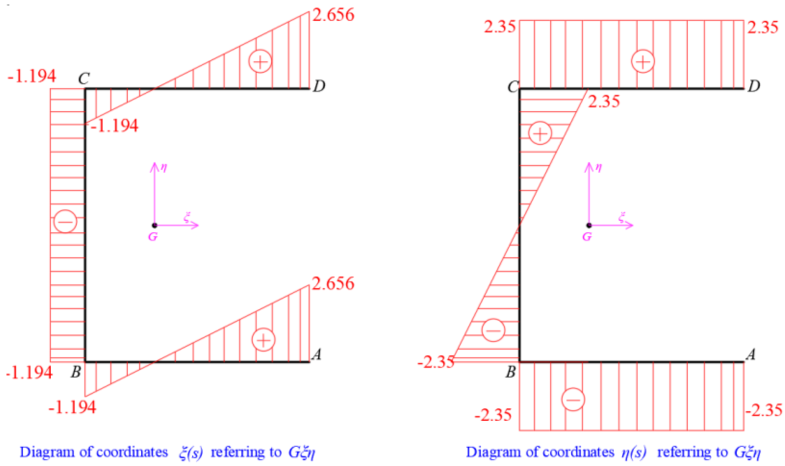

Fig. 2: Diagrams of coordinate functions (s) and (s), relative to the principal gravity Cartesian reference system $G\xi\eta z$

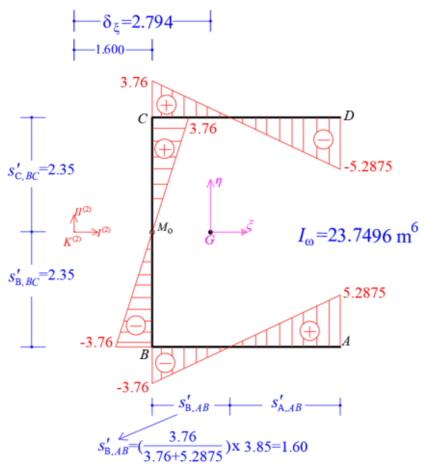

Fig. 3: The exact sectorial coordinates $\omega(s)$ with respect to the pole K and based on the principal start point M_o of the thin-walled open cross-section

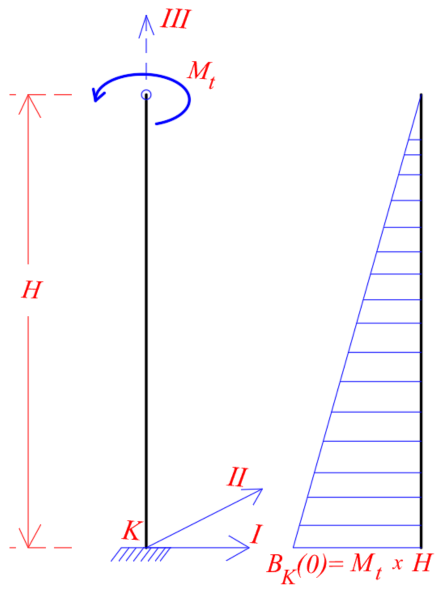

Fig. 5: The bi-moment diagram B_K of the core, due to torsional moment M_t

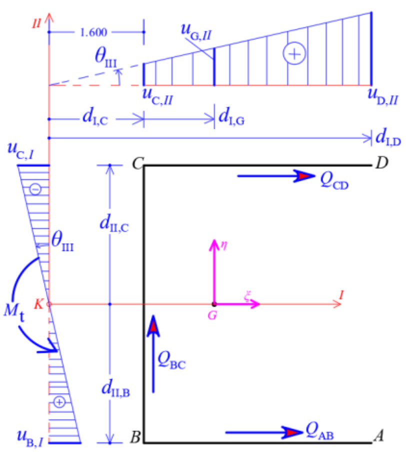

Fig. 4: Angle θ_{III} of the core and the displacement diagrams due to external torsional moment M_t around III – axis of the Cartesian principal elastic reference system $K(I, II, III)$

$$\sigma_z(0,i) = -\omega_i(s)\cdot\frac{B_K(0)}{I_\omega} \quad for\ i = A, B, C, D \tag{1}$$

Hence, the normal stresses $\sigma_z(0,i)$ that have been developed on the cross-section of the core-basis, and are parallel to the vertical III-axis, namely $z = 0$, due to bi-moment $B_K(0)$, are given from the following relationship by the Vlasov torsion theory, (Vlasov, 2022; Makarios and Athanatopoulou, 2022), according to Fig. 6.

Afterward, we can write the following basic equations:

1) The core is rotated per angle θ_{III} around the elastic center K, while the angle is given, generally, by the following relationship:

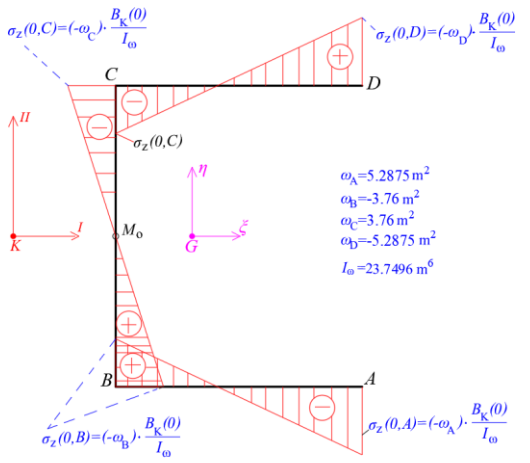

Fig. 6: Diagram of normal stresses $\sigma_z(0,i)$ on the core basis cross-section due to the bi-moment B_K

$$\theta_{III} = \frac{M_t}{k_{\theta,III}} \tag{2}$$

Therefore, if we know the equivalent torsional-warping stiffness $k_{\theta,III}$ of the core (considering the torsional-warping phenomenon), then we calculate the angle θ_{III}.

This phenomenon is called "bend_from_torsion" and is absolutely different from the classical pure torsion according to Saint-Venant torsion theory. Moreover, we consider that the core has fixed-foundation, while has height $H = 4.00\ m$:

2) We work at the principal Cartesian elastic reference system *K(I, II, III)* of Fig. 4 and using the kinematic conditions, the horizontal displacement $u_{c,i}$ of the leg *CD*, along the principal elastic *I*-axis is given:

$$u_{C,I} = -d_{II,C} \cdot \theta_{III} \tag{3}$$

Similarly, the horizontal displacement $u_{B,I}$ of the leg *BA*, along the principal elastic *I*-axis is equal:

$$u_{B,I} = \left(-d_{II,B}\right) \cdot \theta_{III} \tag{4}$$

3) Hence, the two shear forces Q_{AB} and Q_{CD}, which are developed at the top cross-section of the two legs *AB* and *CD* of the core, are given, respectively:

$$Q_{CD} = u_{C,I} \cdot k_{CD} \tag{5}$$

$$Q_{AB} = u_{B,I} \cdot k_{AB} \tag{6}$$

where, k_{AB}, k_{CD} and are the lateral, translational stiffness (in *kN/m*) of the legs *AB* and *CD* of the core. It is worth noting that according to structural analysis, the lateral, translational stiffness of a cantilever, that has cross-section *e.l* is given by the following relationship (considering both, the virtual work due to bend moments and the virtual work due to shear forces), Fig. 7:

$$k_{AB} = \frac{3E \cdot I_{\eta'}}{H^3 + \left(3E \cdot I_{\eta'} \cdot H / \left(G \cdot A_S\right)\right)} \tag{7}$$

where:

$G = E / \left[2(1+v)\right], v =$ the Poisson ratio $I_{\eta'} = e \cdot l^3 / 12$ and $A_s = 0.85(e \cdot l)$ with A_s as the effective shear area of the cross-section of the examined leg:

4) The shear force Q_{AB} that is developed at the top of the leg *AB* gives at the base of the self-leg the flexural moment $M_{\eta',AB}$:

$$M_{\eta',AB} = Q_{AB} \cdot H \tag{8}$$

Similarly, for the leg *CD*:

$$M_{\eta',CD} = Q_{CD} \cdot H \tag{9}$$

5) On the other hand, at the base cross-section $(z = 0)$, the normal stresses $\sigma_z\ (0,i)$ of the legs *AB* and *CD* due to the torsion-warping phenomenon according to Vlasov Torsion Theory are given in Fig. 6. However, the same normal stresses $\sigma_z\ (0,i)$ are connected with the flexural moment of the leg via the Bernoulli Bending Theory. Hence, for leg *AB*, at corner *B* (since examined always the corner that has the minimum magnitude of sectorial coordinate $\omega(A)$ or $\omega(B)$), the equivalent flexural moment $M_{\eta',AB}$ of this leg is given according to Bernoulli bending theory:

$$M_{\eta',AB} = \sigma_Z\left(0,B\right) \cdot \frac{I_{\eta'}}{S'_{B,AB}} \Rightarrow M_{\eta',AB} = \left(-\omega_B\right) \cdot \frac{B_K\left(0\right)}{I_\omega} \cdot \frac{I_{\eta'}}{S'_{B,AB}} \tag{10}$$

where, $\sigma_z\ (0,B)$ is taken from Fig. 6, the bi-moment $B_K(0)$ at the base $(z = 0)$ of the core is $B_K(0) = M_t.H$ and $S'_{B,AB}$ is the distance between corner *B* and the neutral axis of leg *AB* from the diagram of sectorial coordinates $\omega(s)$, Fig. 3.

Summarized, we are written the following useful equations for the leg *CD*:

$$u_{C,I} = -d_{II,C} \cdot \theta_{III} \tag{11}$$

$$Q_{CD} = k_{CD} \cdot u_{C,I} \tag{12}$$

$$M_{\eta'.CD} = Q_{CD} \cdot H \tag{13}$$

$$M_t = k_{\theta,III} \cdot \theta_{III} \tag{14}$$

$$B_K(0) = M_t \cdot H \tag{15}$$

$$M_{\eta',CD} = (-\omega_C) \cdot \frac{B_K(0)}{I_\omega} \cdot \frac{I_{\eta'}}{s^{'}_{C,CD}} \tag{16}$$

Inserting Eqs.11-12 into Eq. 13 we get:

$$M_{\eta',CD} = -d_{II,C} \cdot \theta_{III} \cdot k_{CD} \cdot H \tag{17}$$

Inserting Eq. 14 into Eq. 15 we get:

$$B_K(0) = k_{\theta,III} \cdot \theta_{III} \cdot H \tag{18}$$

Finally, by inserting Eqs. 17-18 into Eq. 16 we get:

$$-d_{II,C} \cdot \theta_{III} \cdot k_{CD} \cdot H = (-\omega_C) \cdot \frac{k_{\theta,III} \cdot \theta_{III} \cdot H}{I_\omega} \cdot \frac{I_{\eta'}}{s^{'}_{C,CD}} \tag{19}$$

The torsional-warping stiffness $k_{\theta,III}$ of the core is produced from Eq. 19 as the following relationship:

$$k_{\theta,III} = \frac{-d_{II,C} \cdot I_\omega \cdot s^{'}_{C,CD}}{(-\omega_C) \cdot I_{\eta'}} \cdot k_{CD} \tag{20}$$

Afterward, inserting the values of the core parameters, and considering that the core material is concrete C30/37, thus $E = 33\ GPa$, we get:

$$k_{CD} = \frac{3E \cdot I_{\eta'}}{H^3 + (3E \cdot I_{\eta'} \cdot H / (G \cdot A_S))} = \frac{141{,}239{,}896.88}{4^3 + (40.107952)} = 1{,}356{,}667.85 kN/m$$

$$k_{AB} = 1{,}356{,}667.85 kN/m$$

because:

$$I_{\eta'} = \frac{0.30 \cdot (3.85)^3}{12} = 1.426666 m^4$$

$$G = E / [2(1+v)] = 33{,}000{,}000. / [2(1+0.15)] = 14{,}347{,}826.09 kN/m^2$$

$$G \cdot A_S = 14{,}347{,}826.09 \cdot (0.85 \cdot 0.30 \cdot 3.85) = 14{,}085{,}978.26 kN$$

$$3E \cdot I_{\eta'} = 3 \cdot 33{,}000{,}000. \cdot 1.426666 = 40.107952 kNm^2$$

$$(3E \cdot I_{\eta'}) \cdot H / (G \cdot A_S) = (141{,}239{,}896.88) \cdot 4 / (14{,}634{,}782.61) = 40.107952 m^3$$

Hence, the torsional-warping stiffness $k_{\theta,III}$ is equal:

$$k_{\theta,III} = \frac{-d_{II,C} . I_\omega . s^{'}_{C,CD}}{(-\omega_C) . I\eta'} . k_{CD} = \frac{-94.9984}{-5.364264} . (1{,}356{,}667.85) = 24{,}025{,}900.89\ kNm$$

because:

$$s'_{C,CD} = s'_{B,AB} = -1.60 m$$

$$-d_{II,C} \cdot I\omega \cdot s'_{C,CD} = (-2.35) \cdot (237496) \cdot (-1.60) = -94.9984 m^8$$

and:

$$(-\omega_C) \cdot I_{\eta'} = (-3.76) \cdot 1.426666 = -5.364264 m^6$$

Hence, if we use as a base the leg *AB*, then the torsional-warping stiffness is $k_{\theta,III} = 24{,}025{,}900.89 kNm$

Hence, for symmetry reasons, if we use as a base the leg *CD*, then the torsional-warping stiffness is $k_{\theta,III} = 24{,}025{,}900.89 kNm$.

The final, equivalent torsional-warping stiffness of this core is always a mean value of the two end legs of the core $k_{\theta,III}$ = 24, 025, 900.89 *kNm*.

It is worth noting that Eq. 20 gives the torsional-warping stiffness $k_{\theta,III}$ of this particular core that has shape ⊏. For each core shape, a similar procedure must be re-written with reference to the two lateral, principal translational stiffnesses k_I and k_{II} of the core, these are given as follows:

I. Lateral Principal Translational Stiffness k_I

$$k_I = \frac{3E \cdot I_\eta}{H^3 + (3E \cdot I_\eta \cdot H / (G \cdot A_S))} = \frac{606{,}329{,}460}{4^3 + (53.4590)} = 5{,}162{,}051.95 kN/m$$

where:

$$I_\eta = 6.12454 m^4$$

$$G = E / [2(1+\upsilon)] = 33{,}000{,}000 / 2(1+0.15) = 14{,}347{,}826.09 kN/m^2$$

$$G. A_s = 14{,}347{,}826.09. (0.85.3.75) = 45{,}367{,}826.1 kN$$

$$3E.I_\eta = 3.33{,}000{,}000.6.12454 = 606{,}329{,}460, kNm^2$$

$$3E.I_\eta = H / (G.A_s) = 606{,}329{,}460.4 / 45{,}367{,}826.1 = 53.4590 m^3$$

II. Lateral Principal Translational Stiffness k_{II}

$$k_{II} = \frac{3E.I_\xi}{H^3 + (3E.I\xi. H / G.A_s)} = \frac{1{,}524{,}757{,}410.}{4^3 + 134.43513} = 7{,}683{,}908.6 kN/m$$

where:

$$I_\xi = 15.40159 m^4$$

$$G = E./\left[2(1+\upsilon)\right] = 33{,}000{,}000./\left[2(1+0.15)\right] = 14{,}347{,}826.09\,kN/m^2$$

$$G \cdot A_s = 14{,}347{,}826.09 \cdot (0.85 \cdot 3.72) = 45{,}367{,}826.1\ kN$$

$$3E \cdot I_\xi = 3 \cdot 33{,}000{,}000. \cdot 15.40159 = 1{,}524{,}757{,}410.\ kNm^2$$

$$3E \cdot I_\eta \cdot H/(G \cdot A_s) = 1{,}524{,}757{,}410 \cdot 4/45{,}367{,}826.1 = 134.43513\,m^3$$

Hence, with reference to the Cartesian principal elastic reference system *K(I, II, III)* of the first mono-symmetric core, the equivalent lateral stiffness matrix K_{core} of it, is given:

$$K_{core} = \begin{bmatrix} k_I & 0 & 0 \\ 0 & k_{II} & 0 \\ 0 & 0 & k_{\theta,III} \end{bmatrix} = \begin{bmatrix} 5{,}162{,}051.95 & 0 & 0 \\ 0 & 7{,}683{,}908.6 & 0 \\ 0 & 0 & 24{,}025{,}900.89 \end{bmatrix} \quad (21)$$

Torsional-Warping Stiffness of the Asymmetric Core That Examined at (Makarios and Athanatopoulou, (2022) Second Case

We consider the core of Fig. 8, which is a structural member of a single-story RC building that has a height equal to 5.5 meters (H = 5.50 m) and had been published (Makarios and Athanatopoulou, 2022).

In this core does not exist symmetry axis. Noting that this is a core of a staircase and, also, is fixed at its foundation, into the ground. The location of the gravity center G, is shown in Fig. 8, while the orientation of the first principal directions of the cross-section, which is symbolized as ξ-axis is ω_o = -40.00932°. The other, second principal direction is perpendicular to the first and is symbolized as the η−axis, while both principal axes have a common origin the gravity center G. The principal moments of inertia are I_ξ and I_η of the thin-walled open cross-section about the principal directions ξ and η passing through the gravity center G of the cross-section, while the product moment of inertia $I_{\xi\eta}$ of the cross-section is zero. Therefore, after the calculations, the principal moments of inertia I_ξ and I_η have the following values:

$$I_\xi = 4.27656\,m^4,\ I_\eta = 8.04292\,m^4$$

Moreover, the area of the cross-section is A = 3.33 m^2. Furthermore, the steps (*i*-*vi*) of the above-mentioned methodology have been applied, and after two repeats the final position of the local elastic center K has been calculated in distance δ_ξ = -0.50319 m along ξ-*axis* and δ_η = -2.76 m along η-axis, Fig. 9. At the same Fig. 10, we can see the location of the principal start point *Mo (xo, yo)* of the thin-walled open section as well as of the exact sectorial coordinates $\omega(s)$ with respect to the pole K and based on the principal start point M_o of the thin-walled open cross-section. The I-axis, which has the origin of the elastic center K, is the principal elastic axis (that it is parallel with the ξ-axis of the core), and the II-axis, which also has the origin of the elastic center K, is the second principal elastic axis (that it is parallel with the η-axis). The third principal elastic axis is vertical and through from the elastic center K, too. Hence, the three principal elastic Axes *I*, *II*, and *III* create the local principal elastic reference system K(*I*, *II*, *III*) of the core. Finally, the numerical value of the warping moment of inertia I_ω has resulted:

$$I_\omega = 16.39462\,m^6$$

All the above-mentioned calculations (Figs. 8-10) are the necessary spadework that is based on the recent article (Makarios and Athanatopoulou, 2022). Afterward, from this point and below, we are following the steps according to the new present methodology.

In order to calculate the torsional-warping stiffness k_θ, of the core, we consider that it is loaded at its top with an external torsional moment M_t, around the vertical principal elastic III-axis. Due to torsional moment M_t, the thin-walled open cross-section of the core is rotated around its Elastic Center K per angle θ_{III}. In Fig. 11 we can see the rotation angle θ_{III} of the core and the displacement diagrams due to the external torsional moment M_t around III-axis of the Cartesian principal elastic reference system *K(I, II, III)*, using the kinematic conditions of the cross-section (that behaves as diaphragm according to (Vlasov torsion theory). Afterward, we consider that this core is loaded at its top with an external static torsional moment. Also, the bi-moment diagram B_K along the height of the core is given in Fig. 5. Hence, the normal stresses $(0,\iota)$ that have been developed on the cross-section of the core-basis, and are parallel to vertical III-axis, namely z = 0, due to bi-moment (0), are given from the following relationship by the Vlasov torsion theory, (Vlasov, 2020; Makarios and Athanatopoulou, 2022), according to Fig. 11:

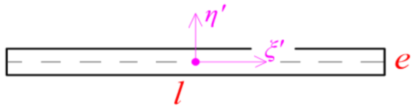

Fig. 7: The cross-section of a leg of the core and the local principal elastic axes ξ', η'

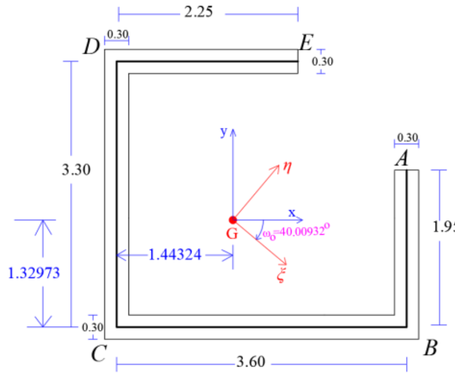

Fig. 8: Geometry of the core

$$\sigma_z(0,i) = -\omega(s).\frac{B_K(0)}{I_\omega} \quad for\, i = A,B,C,D,E \tag{22}$$

1) The core is rotated per angle θ_{III} around the elastic center K and this angle is given by the following general relationship:

$$\theta_{III} = \frac{M_t}{k_{\theta,III}} \tag{23}$$

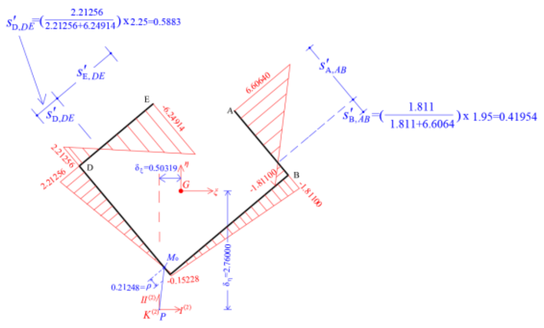

Fig. 9: The exact sectorial coordinates $\omega(s)$ with respect to the pole K and based on the principal start point M_o of the thin-walled open cross-section

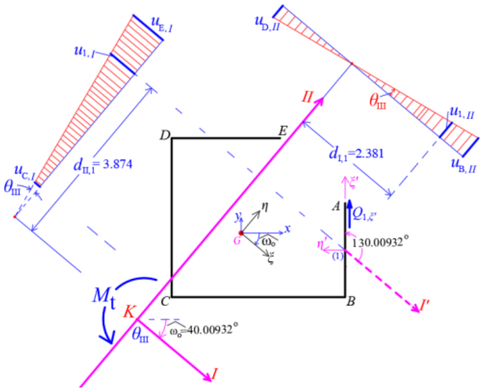

Fig. 10: Angle θ_{III} of the core and the displacement diagrams due to external torsional moment M_t around III-axis of the Cartesian principal elastic reference system $K(I, II, III)$

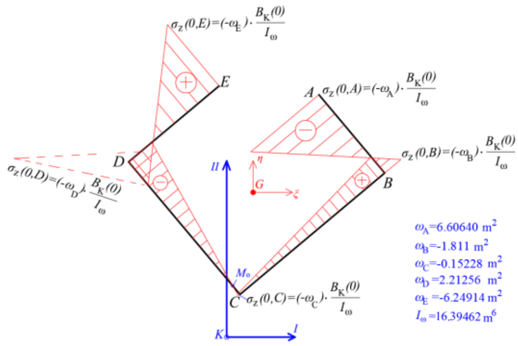

Fig. 11: Diagram of normal stresses σ_z $(0,i)$ on the core basis cross-section due to the bi-moment B_K

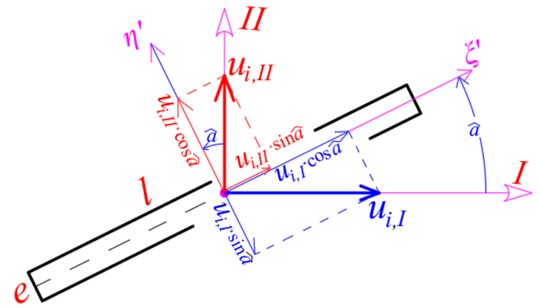

Fig. 12: For i = 1, 2, 3 … local gravity center of a random leg, the transformation of the two displacements $u_{1,}$, and $u_{1,I}$ of leg AB along the two principal elastic axes I and II of a core in displacements along the local principal directions ξ' and η' of the examined leg AB, due to angle θ_{III} of the core, where $\hat{a}$ is the orientation angle of this leg

Therefore, if we know the torsional-warping stiffness $k_{\theta,III}$, of the core (considering the torsional-warping phenomenon), then we calculate the angle θ_{III}. This phenomenon is called "bend_from_torsion" and is absolutely different from the classical pure torsion according to Saint-Venant torsion theory. Moreover, we consider that the core has fixed-foundation, while has height H = 5.50 m.

Here, we consider that point (1) is the local gravity center of the leg AB, has two horizontal displacements $u_{1,I}$ and $u_{1,II}$, along the two principal elastic axes I and II of the core, Fig. 10.

Next, using the local rotation matrix, we can transform these displacements in the local displacements along the local principal directions ξ' and η' of the examined cross-section of leg AB as follows, where $\hat{\alpha}$ is the orientation angle of this leg, Fig. 12:

$$\begin{Bmatrix} u_i,\xi' \\ u_i,\eta' \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_i,_I \\ u_i,_{II} \end{Bmatrix} \tag{24}$$

2) We work at the principal elastic reference system $K(I, II, III)$ of Fig. 10 and using the kinematic conditions, the horizontal displacement $\mu_{1,I}$ of the local gravity center (1) of leg AB, along the principal elastic I-axis is given:

$$u = -d_{II,1}.\theta_{III} \tag{25}$$

And the horizontal displacement $u_{1,II}$, of the local gravity center (1) of leg AB, along the principal elastic II-axis is equal:

$$u_{1,II} = -d_{I,1}.\theta_{III} \tag{26}$$

Therefore, the displacements $u_{1,\xi'}$ and $u_{1,\eta'}$ of the gravity center of the examined leg AB are given at the local principal axes ξ' and η' as:

$$\begin{Bmatrix} u_{1,\xi'} \\ u_{1,\eta'} \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} -d_{II,1}.\theta_{III} \\ d_{I,1}.\theta_{III} \end{Bmatrix} = \begin{Bmatrix} -d_{II,1}.\theta_{III}.\cos\alpha + d_{I,1}.\theta_{III}.\sin\alpha \\ d_{II,1}.\theta_{III}.\sin\alpha + d_{I,1}.\theta_{III}.\cos\alpha \end{Bmatrix} \quad (27)$$

3) Hence, the shear force $Q_{1,\xi'}$, which is developed at the top cross-section of the leg *AB* of the core, is given:

$$Q_{1,\xi'} = k_{1,\xi'} \cdot u_{1,\xi'} \quad (28)$$

where, k_1,ξ' is the translational stiffness (in *kN/m*) of the leg *AB* of the core. It is worth noting that according to Structural Analysis, the lateral, translational stiffness of a cantilever, that has cross-section *e.l* is given by the following relationship (considering both, the virtual work due to bend moments and the virtual work due to shear forces), Fig. 7:

$$k_{1,\xi'} = \frac{3E \cdot I_{1,\eta'}}{H^3 + \left(3E \cdot I_{1,\eta'}.H / \left(G \cdot A_s\right)\right)} \quad (29)$$

where:

$$G = E/2(1+\upsilon), \upsilon = \text{Poisson Ration}, I_{1,\eta'} = e.l^3/12 \quad A_s = 0.58(e.l)$$

4) The sheer force $Q_{1\xi'}$, that is developed at the top of the leg *AB* gives at the base of the self-leg the flexural moment $M_{1,\eta'}$:

$$M_{1,\eta'} = Q_{1,\xi'} \cdot H \quad (30)$$

5) On the other hand, at the base cross-section ($z = 0$), the normal stresses $\sigma_z(0,i)$ of the examined leg *AB* due to torsion-warping phenomenon according to Vlasov Torsion Theory is given in Fig.11. However, the same normal stresses $\sigma_z(0,i)$ are connected with the flexural moment of the leg via the Bernoulli bending theory

Hence, for the leg *AB*, at corner *B* (since examined always the corner that has the minimum magnitude of sectorial coordinate $\omega(A)$ or $\omega(B)$), the equivalent flexural moment $M_{1,\eta'}$ of this leg is given according to Bernoulli Bending Theory:

$$M_{1,\eta'} = \sigma_z(0,B).\frac{I_{1,\eta'}}{s'_{B,AB}} \Rightarrow \quad M_{1,\eta'} = (-\omega_B).\frac{B_K(0)}{I_\omega}.\frac{I_{1,\eta'}}{s'_{B,AB}} \quad (31)$$

where, $\sigma_z(0,B)$ is taken from Fig. 11, the bi-moment $B_K(0)$ at the base ($z = 0$) of the core is $B_K(0) = M_t.H$ and $s'_{B,AB}$ is the distance between corner *B* and neutral axis of leg *AB* from the diagram of sectorial coordinates $\omega(s)$, Fig. 10.

Summarized, we have written the following useful equations for the leg *AB*:

$$u_{1,\xi'} = -d_{II,1} \cdot \theta_{III} \cdot \cos\alpha + d_{I,1} \cdot \theta_{III} \cdot \sin\alpha \quad (32)$$

$$Q_{1,\xi'} = -k_{1,\xi'} \cdot \mu_{1,\xi'} \quad (33)$$

$$M_{1,\eta'} = Q_{1,\xi'}.H \quad (34)$$

$$M_t = k_{\theta,III} \cdot \theta_{III} \quad (35)$$

$$B_K(0) = M_t \cdot H \quad (36)$$

$$M_{1,\eta'} = (-\omega_B) \cdot \frac{B_K(0)}{I_\omega} \cdot \frac{I_{1,\eta'}}{s'_{B,AB}} \quad (37)$$

Inserting Eqs. 32-33 into Eq. 34 we get:

$$M_{\eta',1} = k_{1,\xi}' \cdot \left(-d_{II,1} \cdot \theta_{III} \cdot \cos\alpha + d_{I,1} \cdot \theta_{III} \cdot \sin\alpha\right) \cdot H \quad (38)$$

Inserting Eq. 35 into Eq. 36 we get:

$$B_K(0) = k_{\theta,III} \cdot \theta_{III} \cdot H \quad (39)$$

Finally, inserting Eqs. 38-39 into Eq. 37 we get:

$$k_{1,\xi'} \cdot \left(-d_{II,1}.\theta_{III} \cdot \cos\alpha + d_{I,1} \cdot \theta_{III} \cdot \sin\alpha\right) \cdot H = (-\omega_B) \cdot \frac{k_{\theta,III} \cdot \theta_{III} \cdot H}{I_\omega} \cdot \frac{I_{1,\eta'}}{s'_{B,AB}} \quad (40)$$

The torsional-warping stiffness $k_{\theta,\mathrm{III}}$, of the core, is produced from Eq. 40 as the following relationship:

$$k_{\theta,III} = \frac{\left(-d_{II,1} \cdot \cos\alpha + d_{I,1} \cdot \sin\alpha\right) \cdot I_\omega \cdot s'_{B,AB}}{(-\omega_B) \cdot I_{1,\eta'}} \cdot k_{1,\xi'} \quad (41)$$

Afterward, inserting the values of the core parameters, and considering that the core material is concrete C30/37, thus E = 33GPa, we get:

$$k_{1,\xi'} = \frac{3E \cdot I_{1,\eta'}}{H^3 + \left[3E.I_{1,\eta'} \cdot H / \left(G \cdot A_s\right)\right]} = \frac{18{,}351{,}815.63}{5.5^3 + [140147537]} = 101{,}659.42 \ kN/m$$

because:

$$I_{1,\eta'} = \frac{0.30.1.95^3}{12} = 0.185372\,m^4$$
$$G = E/\left[2(1+\upsilon)\right] = 33{,}000{,}000/\left[2(1+0.15)\right] = 14{,}347{,}826.09\,kN/m^2$$
$$G \cdot A_s = 14{,}347{,}826.09.\,(0 \cdot 85 \cdot 0 \cdot 30 \cdot 1 \cdot 95) = 7{,}134{,}456.52\,kN$$

$$3E.I_{1,\eta'} = 3 \cdot 33{,}000{,}000 \cdot 0.185372 = 18{,}351{,}815.63\ kNm^2$$
$$3E.\,I_{1,\eta'}\,.\,H/(G \cdot A_s) = 18{,}351{,}815.63 \cdot 5.5/7{,}134{,}456.52 = 14.147537\,m^3$$

Hence, the torsional-warping stiffness $k_{\theta,III}$ is equal:

$$k_{\theta,III} = \frac{(-d_{II,1} \cdot \cos\alpha + d_{I,1} \cdot \sin\alpha) \cdot I_\omega \cdot s'_{B,AB}}{(-\omega_B) \cdot I_{1,\eta'}} \cdot k_{1,\xi'} =$$
$$\frac{29.674921}{0.335709} \cdot 101{,}659.42 = 8{,}986{,}175.37\,kNm/rad$$

because:

$$d_{I,1} = 2.381,\ d_{II,1}\ 3.874$$

$$(-d_{II,1} \cdot \cos\alpha + d_{I,1} \cdot \sin\alpha) \cdot I_\omega \cdot s'_{B,AB} =$$
$$[-3.874 \cdot \cos(130.00932) + 2.381 \cdot \sin(130.00932)]$$

$$(16.39462) \cdot (0.41954) = 29.674921\,m^8$$

and:

$$(-\omega_B) \cdot I_{\eta'} = (1.81100) \cdot 0.185372 = 0.335709\,m^6$$

Hence, if we use as a base the leg *AB*, then the torsional-warping stiffness is: $k_{\theta,III} = 8{,}986{,}175.37$ *kNm* in a similar way, if we use as a base the leg *DE*, then the torsional-warping stiffness is $k_{\theta,III} = 8{,}767{,}901.00$ *kNm.*

The final equivalent torsional-warping stiffness of this core is always a mean value of the two end legs of the core $k_{\theta,III} = 8{,}877{,}038.19$ *kNm.*

With reference to the two lateral, principal translational stiffnesses k_I and k_{II} of the core, these are given as follows:

I. Lateral principal Translational Stiffness k_I

$$k_I = \frac{3E \cdot I_\eta}{H^3 + (3E \cdot I_\eta \cdot H/(G \cdot A_s))} = \frac{796{,}249{,}0.80.}{5.5^3 + (107.83565)} = 2{,}903{,}786.10\ kN/m$$

where:

$$I_\eta = 8.04292\,m^4$$

$$G = E/\left[2(1+\upsilon)\right] = 33{,}000{,}000./\left[2(1+0.15)\right] = 14{,}347{,}826.09\,kN/m^2$$
$$G \cdot A_s = 14{,}347{,}826.09 \times (0.85 \cdot 3.33) = 40{,}611{,}521.75\,kN$$
$$3E \cdot I_\eta = 3 \cdot 33{,}000{,}000. \times 8.04292 = 796{,}249{,}080.\ kNm^2$$

$$3E \cdot I_\eta \cdot H/(G \cdot A_s) = 796{,}249{,}080 \cdot 5 \cdot 5/40{,}611{,}521.75 = 107.83565\ m^3$$

II. Lateral Principal Translational Stiffness k_{II}

$$k_{II} = \frac{3E \cdot I_\xi}{H^3 + (3E \cdot I_\xi \cdot H/(G \cdot A_s))} = \frac{423{,}379{,}440.}{5.5^3 + (57.338086)} = 1{,}892{,}510.84\ kN/m$$

where:

$$I_\xi = 4.27656\,m^4$$

$$G = E/\left[2(1+\upsilon)\right] = 33{,}000{,}000./\left[2(1+0.15)\right] = 14{,}347{,}826.09\ kN/m^2$$
$$G \cdot A_s = 14{,}347{,}826.09 \times (0.85 \cdot 3.33) = 40{,}611{,}521.75\,kN$$
$$3E \cdot I_\xi = 3 \cdot 33{,}000{,}000. \times 4.27656 = 423{,}379{,}440.\,kNm^2$$

$$3E \cdot I_\eta \cdot H/(G \cdot A_s) = 423{,}379{,}440. \times 5.5/40{,}611{,}521.7 = 57.338086\ m^3$$

Hence, with reference to the Cartesian principal elastic reference system *K(I, II, III)* of the second asymmetric core, the equivalent lateral stiffness matrix K_{core} of it, is given:

$$K_{\text{core}} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_{II} & 0 \\ 0 & 0 & k_{\theta,III} \end{bmatrix} = \begin{bmatrix} 2{,}903{,}786.10 & 0 & 0 \\ 0 & 1{,}892{,}510.84 & 0 \\ 0 & 0 & 8{,}877{,}038.19 \end{bmatrix}$$

Results and Discussion

The results of the present analysis permit the use of an equivalent column located at the elastic center *K* of the core. This column must be two lateral bending-shear stiffness for clear moving along the two horizontal principal axes and an equivalent torsional stiffness for the rotation of the core about the vertical axis, which pass-through point *K*. Also, this column must have axial-stiffness zero. Additionally, at the center of gravity of each leg of the core, a column with axial-stiffness (but with very small moment of inertia) of the leg of the core must be inserted. More details about it, are now in progress and it is beyond out of the target of the present article.

Conclusion

In order to simulate documented RC core with a thin-walled open cross-section, the present methodology has proposed to use an ideal-equivalent column that has to be located on the elastic center *K* of the core. This equivalent-ideal column must be provided with a diagonal, lateral-stiffness matrix that represents equivalently the properties of the real core. The present article has given a solution and has presented the following two numerical examples of cores: (a) The first example is a monosymmetric core and (b) The second example is an asymmetric core. In order to calculate the above-mentioned diagonal, lateral stiffness matrix of the reinforced concrete cores that have thin-walled open cross-sections, a new analytical methodology, which combines the Vlasov torsion theory with the Bernoulli bending theory, has been presented. As the basis of the calculations we have used the principal elastic reference system of the RC core, the principal start point of the cross-section, the exact sectorial coordinates as well as, the warping moment of inertia of the open cross-section of the examined core. All these have been analytically known according to recent work (Makarios and Athanatopoulou, (2022). Furthermore, the two above-mentioned theories (Vlasov and Bernoulli) are together combining and in the end, the equivalent torsional-warping stiffness of the core has resulted. We ascertain that due to the fact that the lateral-stiffness matrix K_{core} is a diagonal matrix, there is uncoupling between the three degrees of freedom (u_I, u_{II} and θ_{III} of the elastic center) of the core. Next, we consider the loading vector P that has consisted of two forces P_I and P_{II} along the two principal axes, respectively, and a torsional moment M_{III} around the III-axis, with reference to these three degrees of freedom of the point K. If a core is loaded with the lateral-loading vector P then the balance equation is written:

$$K_{core}\,u = P \Rightarrow \begin{bmatrix} k_I & 0 & 0 \\ 0 & k_{II} & 0 \\ 0 & 0 & k_{\theta,III} \end{bmatrix} \begin{bmatrix} \mu_I \\ \mu_{II} \\ \theta_{III} \end{bmatrix} = \begin{bmatrix} P_I \\ P_{II} \\ M_{III} \end{bmatrix} \quad (42)$$

Hence, the three degrees of freedom u_I, u_{II}, and θ_{III} of the elastic center of the core are uncoupled. From this last property the following conclusions have been resulted:

a. If a lateral static force P_I (having the same orientation as I-axis) is applied on the elastic center K of a thin-walled open cross-section, then the cross-section is moving parallel to itself along the I-axis, while the displacement along the II-axis is null. Moreover, the rotation around the z-axis of the cross-section is null, too. Hence, the I-axis is called the principal I-Axis of the cross-section
b. If a lateral static force P_I (having the same orientation as II-axis) is applied on the Elastic Center K of a thin-walled open cross-section having, then the cross-section is moving parallel to itself along the II-axis, while the displacement along the I-axis is null. Moreover, the rotation around the z-axis of the cross-section is null, too. Hence, the II-axis is called the principal II-Axis of the cross-section
c. If there is an axis of symmetry at the cross-section, then it is always the principal axis of the cross-section
d. If a torsional moment, M_{III} (about the vertical III-axis) is applied on the elastic center K of a thin-walled open cross-section of core, then the horizontal displacements u_I and u_{II} of the elastic center K are null, hence the point K is called as Center of twist of the cross-section
e. If a lateral static force P is applied on the elastic center K of a thin-walled open cross-section (having random orientation), then the rotation θ_{III} about the vertical III-axis is null, hence the point K is called the center of the bending of the cross-section
f. For a random lateral static force P that is acting on any point of the thin-walled open cross-section, and if we consider that the rotation θ_{III} (about the vertical axis) of the cross-section has been fixed, then the equivalent base shear-force of the cross-section is passed through point K. Hence, the point K is the center of Shear of the thin-walled open cross-section
g. The Shear Forces (recovery elastic forces) Q_ξ, Q_η are dependent on the moments of inertia I_ξ and I_η of the core, but these forces are acting on the elastic center K of the cross-section
h. Last but not least, in order to check the reliability of the results of various analysis software, the proposed procedure can be used as a benchmark analysis method of RC cores. It is worth noting that this lateral stiffness matrix K_{core} can be used as it is, directly, at the single-story building, while multi-story buildings need more processing that is out of the target of the present article

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