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Some Conditions for P-Solubility of Finite Groups

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Abstract: A subgroup H of a group G is c-subnormal in G if G has a subnormal subgroup T such that HT=G and T 3 H ⊆ HG.[1] Using this concept, in Jaraden obtain[1] some new conditions for solubility of a finite group are given. Here we obtain local versions of these results.

Key words: Finite group, p-soluble group, maximal subgroup, normal index, c-subnormal subgroup

INTRODUCTION

All groups that we consider are finite. Let M be a maximal subgroup of a group G. Then normal index |G: M|n of M in G is equal to |H/K| where H/K is a chief factor of G such that K ⊆ M and H ⊄ M (we note that every two chief factors with such property are isomorphic). This concept was introduced by Deskins[2] where the following nice result was proved: A group G is soluble if and only if for every its maximal subgroup M it is true that |G: M| = |G: M|n. Local versions of this result were obtained by many researchers[3-6]. In Wang[7], analyzing the concept of normal index, introduced the following important concept: A subgroup H of a group G is said to be c-normal if there exists a normal subgroup T of G such that HT = G and T 3 H ⊆ HG (where HG is the intersection of all G-conjugates of H, i.e., the unique largest normal subgroup of G contained in H). Using this concept Wang obtained[7] several new interesting results on soluble and supersoluble groups. The concept of c-normal subgroup was used and analyzed. In particular, by Jaraden[1] the following its generalization was considered.

Definition: A subgroup H of a group G is said to be c-subnormal in G if there exists a subnormal subgroup T such that HT = G and T 3H ⊆ HG.

Using this concept, by Jaraden[1] obtained some new conditions for solubility of a group were obtained. Here we prove the following theorems.

Theorem 1: A group G is p-soluble if and only if every maximal subgroup M with p | |G: M|n is c-subnormal in G.

Theorem 2: A group G is p-soluble if and only if it has a p-soluble maximal subgroup M such that either p| |G: M|n or M is c-subnormal in G.

PRELIMINARIES

Notation is standard[8-10].

We shall need the following well known facts about subnormal subgroups.

Lemma 1: Let G be a group, H be a subgroup of G. Then the following statements hold:

- If H is subnormal in G and M ≤ G, then H 3 M is subnormal in M.
- If K ⊲ G and H is subnormal in G, then HK/K is subnormal in G/K.

Lemma 2: Let L be a minimal normal subgroup of a group G and T be a subnormal subgroup of G. Then L ⊆ NG(T).

The following useful lemma was proved by Beildleman and Spencser[4].

Lemma 3: Let M be a maximal in G subgroup, N/G and N_M. Then |G: M|n = |G/N: M/N|n.

Lemma 4: (Frattini argument). Let N be a normal subgroup of a group G and Np be a Sylow p-subgroup of N. Then G = NNG(Np).

Recall that a primitive group is a group G such that for some maximal subgroup U of G, UG =1.

A primitive group is of one of the following types (see [8; A,(15.2)]):

- Soc(G), the socle of G is an abelian minimal normal subgroup of G, complemented by U.
- Soc(G) is a non-abelian minimal normal subgroup of G.
- Soc(G) is the direct product of the two minimal normal subgroups of G which are both non-abelian and complemented by U.

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Lemma 5: Let M be a maximal subgroup of G with MG = 1, where G is a primitive group of type 2[11]. Let R = Soc(G) be the socle of G. If R \ M = 1, then M is a primitive group of type 2 and the simple component of R is isomorphic to a section of a simple component of Soc(M).

We shall also need the following observations on c-subnormal subgroups.

Lemma 6: Let G be a group and H a subgroup of G[1]. Then the following statements are true:

- If H is c-subnormal in G and H ≤ K ≤ G, then H is c-subnormal in K;
- Let K/G and K ≤ H. Then H is c-subnormal in G if and only if H/K is c-subnormal in G/K.
- If K/G and H is c-subnormal in G, then HK/K is c- subnormal in G/K.

PROOFS OF THEOREM 1 AND 2

Proof of Theorem 1: First assume that G is a p-soluble group. Let M be a maximal subgroup of G. Assume that p | |G: M|n. Let H/MG be a chief factor of G. Then p | |H/MG| and so H/MG is an abelian p-group. Hence H 3 M = MG. Thus M is c-subnormal in G.

Now assume that every maximal subgroup M of G with p | |G: M|n is c-subnormal in G. We shall show that G is p-soluble. Assume that it is false and let G be a counterexample with minimal order. Then

- p | |G| (it is evident)
- G is not simple. Indeed, assume that G is simple and let M be a maximal in G subgroup. Then p| |G| = |G:M|n and so by hypothesis M is c-subnormal in G. Let T be a subnormal subgroup of G such that MT = G and T 3 M ⊆ MG = 1. Then |T| = |G: M| = 1, a contradiction. Hence G is not simple.
- If R be a minimal normal subgroup of G, then R =Soc(G) is the unique minimal normal subgroup of G, R is not abelian and p | |R|.

Let H be a non-indentity normal subgroup of G. And let M/H be a maximal subgroup of G/H. Assume p | |G/H: M/H|n. Then in view of Lemma 3, p | |G: M| and so by hypothesis M is c-subnormal in G. Now using Lemma 6, we see that M/H is c-subnormal in G/H. Thus the hypothesis holds for G/H. But |G/H| < |G| and so by the choice of G we conclude that G/H is p-soluble. Since the class of all p-soluble groups is a formation we see that R =Soc(G) is the unique minimal normal subgroup of G. It is clear also that p | |R| and that R is not abelian.

- G has a maximal subgroup M such that R ⊄ M and p | |G: M|.
- Indeed, let Rp be a Sylow p-subgroup of R, P be a Sylow p-subgroup of G such that Rp ⊆ P. Let N = NG(Rp) be the normalizer of Rp in G. Then since Rp = R 3 P ⊲ P, P ⊆ N. Besides since R is not abelian, we have N ≠ G. Now let as choose a maximal subgroup M of G such that N ⊆ M. Then of course p | |G: M|. We note also that R ⊄ M. Indeed, by Frattini argument, G = RN. But N ⊆ M and so R ⊄ M.
- M has a subnormal complement T in G.

Since by (4) R ⊄ M, we have MG = 1 and so p| |R| = |G: M|n. Hence by hypothesis M is c-subnormal in G. Therefore G has a subnormal subgroup T such that TM = G and T 3 M ⊆ MG = 1.

- Final contradiction.

Let L be a minimal subnormal subgroup of G contained in T. Let L^G be the normal closure of L in G. Then L^G ≠ 1 and so R ⊆ L^G. Assume that L ⊄ R. Then by Lemma 1,

L 3 R is a subnormal subgroup of G and 1⊆ L 3 R ⊆ L. Hence L 3 R = 1, since L is a minimal subnormal subgroup of G. By Lemma 2, R ⊆ NG(L). Hence < L, R >= LR = L × R. But then L ⊆ CG(R). Since CG(R) ⊲ G and R ⊆ CG(R). Then R is an abelian group. This contradiction shows that L ⊆ R. Since R is a minimal normal subgroup of G,

R = A1 ×... × At, where A1 ≈ A2 ≈.. ≈ At ≈ A and A is a anon-abelian simple group. Hence L ≈ A. Clearly p divides the order |A| of the group A. Hence p divides the order |L| of the group L. By Lagrange‘s theorem the order |L| of the group L divides the order T of the group T. Hence the prime p divides |T|. We have known that

G = TM and T 3 M = 1. Hence |G| = |T||M| = |G: M||M| and so |T| = |G: M|. But the prime p does not divide the index |G: M| of M in G. Hence p does not divide |T|. This contradiction shows that G is a p-soluble group[4].

The theorem is proved.

Proof of Theorem 2: n view of Theorem 1 we have only to prove the sufficiency. Assume that it is false and let G be a counterexample with minimal order. Then

- G/N is p-soluble for every non-identity normal subgroup N b G.

Indeed, if N ⊄ M, then G/N = MN/N ≈ M/N 3 M is p-soluble. Let N ⊆ M. Then M/N is a p-soluble maximal subgroup of G/N such that either M/N is c-subnormal in G or p ∣ |G/N: M/N|n = |G: M|n. Hence the hypothesis holds for G/N and so G/N is p-soluble by the choice of G since |G/N| < |G|.

- G has unique minimal normal subgroup H which is non-abelian and p ∣ |H| (it directly follows from (1)).
- G has a subnormal subgroup T such that G = TM and T \M = 1.

Since by hypothesis M is p-soluble, then in view of (2) H ⊄ M. Now it is clear that |H| = |G: M|n and so by (2), p ∣ |G: M|n. Hence by hypothesis M is c-subnormal in G. Let T be a subnormal in G subgroup such that TM = G and T 3 M ⊆ MG. But H ⊆ M and so MG = 1. Hence T 3 M = 1. (4) If

1 = H0 ≤ H1 ≤.. ≤. Hn = T = T0 ≤T1 ≤... ≤ Tm = G (1)

is a composition series of G, then every factor T1/T0,..., Tm/Tm−1 is either a group of order p or a p'-group.

It is clear |G: T| = |T1/T0||T2/T1|... |Tm/Tm−1|. Now we consider the following series

1=T03M≤T13M ≤... ≤ Tm−1 3 M ≤ Tm 3 M = M (2)

Evidently Ti−13 M ⊲ Ti 3 M for all i = 1, 2,...,m. Note also that

|(T1 3 M)/(T0 3 M)||(T2 3 M)/(T1 3 M)|... |(Tm 3 M)/(Tm−1 M 3)| = |M| = |G: T| = |T1/T0||T2/T1|... |Tm/Tm−1|.

Since

(Ti 3 M)/(Ti−1 3 M) = (Ti 3 M)/(Ti 3 M)\Ti−1≈ 'Ti−1(Ti 3 M)/Ti−1 ≤ Ti/Ti−1,
|(Ti 3 M)/(Ti−13 M)| ≤ |Ti/Ti−1|

for all i = 1, 2,...,m, a so (Ti 3 M)/(Ti−1 3 M) ' Ti/Ti−1 is a simple group for all i = 1, 2,...,m. Thus series (2) is a composition series of the group M. By hypothesis M is p-soluble. Hence every factor of the series (2) is either a group of order p or a p0-group and so every factor T1/T0, T2/T1,..., Tm/Tm−1 is too.

H\M = 1.

Let H = A1 ×... × At where A1 ≈... ≈ At ≈ A is a non-abelian simple group. Let us consider the following composition series of G:

1 ≤ A1 ≤ A1A2 ≤... ≤ A1A2...At−1 ≤ H
= K0 ≤ K1 ≤... ≤ Kr = G (3)

By Jordan-Holder Theorem [8; I,11.5] there exist indices i1, i2,..., it such that A1 ≈ Hi1/Hi1−1, A1A2/A1 ≈ Hi2/Hi2−1 ,...,H/A1 ...At−1 ≈ Hit/Hit−1. Hence |H| ≤ |T| = |G: M|. But |G: M| = |H: H 3 M| and so H\M = 1.

Final contradiction.

Let A be a composition factor of H. In view of (2), The group G is primitive of type 2 and so by (5) and Lemma 5, A is isomorphic to some section D/L where D ≤ Soc(M). But by hypothesis M is p-soluble and so A is p-soluble. Then H is a p-soluble group and therefore H is a p-group, contrary to (2).
The theorem is proved.

SOME APPLICATIONS

Theorems 1: and 2 have many corollaries. The most important of them we consider in this section.

Corollary 1: A group G is soluble if every its maximal subgroup M is c-subnormal in G[1].

Corollary 2: A group G is soluble if it has a soluble maximal subgroup M which is c-subnormal in G[12].

Corollary 3: A group G is soluble if every its maximal subgroup M is c-normal in G[7].

Corollary 4: A group G is soluble if it has a soluble maximal subgroup M which is c-normal in G[7].

It was proved that for a maximal subgroup M of a group G the following conditions are equivalent[7]:

- M is c-normal in G;
- |G: M| = |G: M|n.

Thus one can obtain from Theorem 1,2 the following known results.

Corollary 5: (W.E. Deskins[2]). A group G is soluble if for every its maximal subgroup M we have |G: M| = |G: M|n.

Corollary 6: (A.Ballester-Bolinches[5]). A group G is p-soluble if for every its maximal subgroup M we have either p | |G: M|n or |G: M| = |G: M|n.

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