Journal of Mathematics & Statistics 1 (3): 172-179, 2005
ISSN 1549-3644

Introduction to the Besov Spaces and Triebel-Lizorkin Spaces for Hermite and Laguerre expansions and some applications.

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Abstract: We introduced new definitions of Besov spaces and Triebel-Lizorkin spaces associated with multidimensional Hermite expansions and multidimensional Laguerre expansions. We showed that the set of p-integrable functions is a Triebel-Lizorkin space with respect to the Gaussian measure and similarly, with respect to the probabilistic Gamma measure. Also, we showed that the Gaussian Sobolev spaces and Laguerre Sobolev spaces are Triebel-Lizorkin spaces, associated with Hermite and Laguerre expansions respectively. We defined Carleson measures with respect to the Gaussian measure and probabilistic Gamma measure. By using maximal functions, related to the Ornstein Uhlenbeck semigroup and Laguerre semigroup, we studied these measures, giving a version of Fefferman's theorem. Finally, we stated relations between Besov spaces and Triebel-Lizorkin spaces.

Key words: Hermite expansions, Laguerre expansions, Fractional Derivate, Potentials spaces, Carleson measures, Besov spaces, Triebel Lizorkin spaces, Meyer's multiplier theorem, Littlewood Paley theory

INTRODUCTION

Lebesgue $L^p(\lambda_d)$ spaces, Hardy $H^p(\lambda_d)$ spaces, Sobolev $L^p_\alpha(\lambda_d)$ spaces, Lipschitz Λ_α spaces and BMO spaces are considered in Harmonic analysis with many different applications (where λ_d is the Lebesgue measure on $\mathbb{R}^d$).

From the original definitions of these spaces, it may not appear that they are related, but there are various unified approaches to their study.
Littlewood Paley theory, Calderon's formula, Fractional Derivates and Atomic Decomposition allow us to consider general functions spaces.

These general classes of spaces are the Besov $\dot{B}^{\alpha,q}_p(\lambda_d)$ spaces and the Triebel-Lizorkin $\dot{F}^{\alpha,q}_p(\lambda_d)$ spaces. These spaces were studied[1] and they are defined as follows.

Let us consider φ a rapidly decreasing function so that $\hat{\varphi}\subset\{\xi:\frac{1}{2}\le|\xi|\le 2\}$ and $|\hat{\varphi}(\xi)|\ge c>0$ if $\frac{3}{5}\le|\xi|\le\frac{5}{3}$. For $\alpha\in\mathbb{R}$, $p\neq\infty$, $0<p,q\le\infty$ and f a tempered distribution, the homogeneous Triebel Lizorkin $\dot{F}^{\alpha,q}_p(\lambda_d)$ space is the set of all such f for which

$$\|f\|_{\dot{F}^{\alpha,q}_p}=\left\|\left(\sum_{v\in Z}(2^{v\alpha}\,|\varphi_{2^{-v}}*f\,|)^q\right)^{1/q}\right\|_{p,\lambda_d}<\infty,$$

and for the same indices including $p=\infty$, the homogeneous Besov space $\dot{B}^{\alpha,q}_p(\lambda_d)$, is the set of all such f for which

$$\|f\|_{\dot{B}^{\alpha,q}_p}=\left(\sum_{v\in Z}2^{v\alpha}\left\|\varphi_{2^{-v}}*f\right\|^q_{p,\lambda_d}\right)^{1/q}<\infty.$$

Inhomogeneous versions of these spaces, $F^{\alpha,q}_p(\lambda_d)$ and $B^{\alpha,q}_p(\lambda_d)$, are obtained by adding the term $\|f\|_{p,\lambda_d}$.

In this context, the following identifications are known:

i. $L^p(\lambda_d)\sim\dot{F}^{0,2}_p(\lambda_d)\sim F^{0,2}_p(\lambda_d)$ when $1<p<\infty$.

ii. $H^p(\lambda_d)\sim\dot{F}^{0,2}_p(\lambda_d)$ when $0<p\le 1$.

iii. $\dot{L}^p_\alpha(\lambda_d)\sim\dot{F}^{\alpha,2}_p(\lambda_d)$ and $L^p_\alpha(\lambda_d)\sim F^{0,2}_p(\lambda_d)$ when $1<p<\infty$ and $\alpha>0$.

iv. $BMO\sim\dot{F}^{0,2}_\infty(\lambda_d)$. Particularly $f\in BMO$ if and only if $d\mu(x,t)=|(\varphi_t*f)(x)|^2\,dx\frac{dt}{t}$ is a Carleson measure.

By other hand, Triebel-Lizorkin spaces on spaces of homogeneous type have been introduced by Coifman and Weiss[2]. New versions of these spaces have been studied by Han and Yang[3].

Continuous versions of the Triebel-Lizorkin spaces were considered by Gatto and Vagi[4]. In their article they considered (X,δ,σ) a normal space of homogeneous type where δ is a quasidistance and σ is

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an infinite measure such that $\sigma(\{x\})=0$, $\forall x \in X$.

Then, they introduced the Triebel-Lizorkin $F_p^{\alpha,2}(\sigma)$ and showed that $L_\alpha^p(\sigma) \sim F_p^{\alpha,2}(\sigma)$, with $1<p<\infty$, and $0<\alpha<1$, by means the Fractional Derivate D_α^σ operator. This Fractional Derivate was introduced[5].

The main purpose of this article is to introduce news definitions of Besov $\dot{B}_p^{\alpha,q}$ spaces and Triebel-Lizorkin $\dot{F}_p^{\alpha,q}$ spaces associated with Hermite expansions and Laguerre expansions. It is important to notice that these spaces ($\dot{B}_p^{\alpha,q}$ and $\dot{F}_p^{\alpha,q}$) had not been defined previously. We obtained the following identifications:

i. $L^p \sim \dot{F}_p^{0,2}$ when $1<p<\infty$. It is possible by means Littlewood Paley theory.

ii. $\dot{L}_\alpha^p \sim \dot{F}_p^{\alpha,2}$ when $1<p<\infty$, and $0<\alpha<1$. Here, the Fractional Derivate operator plays an important role.

iii. Carleson measures: A version of Fefferman's theorem will be given.

Also, we state relations between Triebel Lizorkin spaces and Besov spaces.

Finally, I wish to express my deep gratitude to Prof. P. Graczyk for his enthusiastic encouragement.

PRELIMINARIES

Let us consider the Gaussian measure $\gamma_d(dx)=\frac{e^{-|x|^2}}{\pi^{d/2}}dx$ with $x\in\mathbb{R}^d$ and the probabilistic Gamma measure $\mu_{d,\lambda}(dx)=\prod_{i=1}^d \frac{x_i^{\lambda_i}e^{-x_i}}{\Gamma(\lambda_i+1)}dx$, with $x\in\mathbb{R}_+^d$ and $\lambda\in(-1,\infty)^d$. Let $\beta=(\beta_1,\ldots,\beta_d)\in\mathbb{N}^d$ be a multi-index, let $\beta!=\prod_{i=1}^d \beta_i!$, $|\beta|=\sum_{i=1}^d \beta_i$, $\partial_i=\frac{\partial}{\partial x_i}$, for each $1\le i\le d$ and $\partial^\beta=\partial_1^{\beta_1}\ldots\partial_d^{\beta_d}$. we consider the Ornstein-Uhlenbeck differential operator and the Laguerre differential operator

$$L=\frac{1}{2}\Delta_x-\langle x,\nabla_x\rangle, \text{ and } \mathfrak{L}_\lambda=\sum_{i=1}^d x_i\partial_i^2+(\lambda_i+1-x_i)\partial_i.$$

Let us consider the normalized Hermite polynomials of order β, in d variables

$$h_\beta(x)=\frac{1}{\left(2^{|\beta|}\beta!\right)^{1/2}}\prod_{i=1}^d(-1)^{\beta_i}e^{x_i^2}\partial_i^{\beta_i}(e^{-x_i^2}),$$

and the normalized Laguerre polynomials,

$$l_\beta^\lambda(x)=\prod_{i=1}^d\frac{e^{x_i}x_i^{-\lambda_i}}{\beta_i!\sqrt{\Gamma(\lambda_i+1)\binom{\beta_i+\lambda_i}{\beta_i}}}\partial_i^{\beta_i}(e^{-x_i}x_i^{\beta_i+\lambda_i}).$$

Then, it is well known that, the Hermite polynomials and Laguerre polynomials are eigenfunctions of L and $\mathfrak{L}_\lambda$ respectively. This is,

(2.1) $Lh_\beta(x)=-|\beta|h_\beta(x)$ and $\mathfrak{L}_\lambda l_\beta^\lambda(x)=-|\beta|l_\beta^\lambda(x)$.

Given a function $f\in L^1(\gamma_d)$ or $f\in L^1(\mu_{d,\lambda})$, its β Fourier-Hermite coefficient and Fourier-Laguerre coefficient are defined by

$$c_\beta^f=\langle f,h_\beta\rangle_{\gamma_d} \text{ and } c_\beta^{f,\lambda}=\langle f,l_\beta^\lambda\rangle_{\mu_{d,\lambda}}.$$

Let us consider, C_n and C_n^λ the closed subspaces of $L^2(\gamma_d)$ and $L^2(\mu_{d,\lambda})$, generate by the linear combinations of $\{h_\beta:|\beta|=n\}$ and $\{l_\beta^\lambda:|\beta|=n\}$ respectively. By the orthogonality of the Hermite and Laguerre polynomials with respect to γ_d and $\mu_{d,\lambda}$, we have the orthogonal decompositions,

$$L^2(\gamma_d)=\oplus_{n=0}^\infty C_n \text{ and } L^2(\mu_{d,\lambda})=\oplus_{n=0}^\infty C_n^\lambda.$$

We denote, J_n and J_n^λ the orthogonal projections of $L^2(\gamma_d)$ onto C_n, and $L^2(\mu_{d,\lambda})$ onto C_n^λ respectively. If f is a polynomial,

$$J_nf=\sum_{|\beta|=n}c_\beta^f h_\beta \text{ and } J_n^\lambda f=\sum_{|\beta|=n}c_\beta^{f,\lambda}l_\beta^\lambda.$$

Let us consider the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\ge0}$ as

$$T_tf(x)=\int_{\mathbb{R}^d}\exp\left(\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}\right)f(y)\gamma_d(dy).$$

Let us consider the Laguerre semigroup

$$M_t^\lambda f(x)=\int_{\mathbb{R}^d}K_t^\lambda(x,y)f(y)\mu_{d,\lambda}(dy),$$

where,

$$\prod_{i=1}^d\frac{(e^{-t}x_iy_i)^{-\lambda_i/2}}{1-e^{-t}}\exp\left(\frac{-e^{-t}(x_i+y_i)}{1-e^{-t}}\right)J_{\lambda_i}\left(\frac{2\sqrt{-e^{-t}x_iy_i}}{1-e^{-t}}\right)$$

and J_{λ_i} denotes the modified Bessel function of the first kind and order λ_i.

Now, by Bochner subordination formula, we define the Poisson-Hermite semigroup $\{P_t\}_{t\ge0}$ and the Poisson Laguerre $\{P_t^\lambda\}_{t\ge0}$ semigroup as

$$P_tf(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}T_{t^2/4u}f(x)du,$$

$$P_t^\lambda f(x)=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-u}}{\sqrt{u}}M_{t^2/4u}^\lambda f(x)du.$$

$\{T_t\}_{t\ge0}$ $\{M_t^\lambda\}_{t\ge0}$ $\{P_t\}_{t\ge0}$ and $\{P_t^\lambda\}_{t\ge0}$ are also strongly continuous semigroups on $L^p(\gamma_d)$ and

$L^p(\mu_{d,\lambda})$, with infinitesimal generator

$$-L, -\mathfrak{L}, (-L)^{1/2} \text{ and } (-\mathfrak{L}_\lambda)^{1/2}$$

respectively. Then by (2.1)

$$T_t h_\beta(x) = e^{-t|\beta|} h_\beta(x) \; P_t h_\beta(x) = e^{-t\sqrt{|\beta|}} h_\beta(x)$$

and similarly,

$$M_t^\lambda h_\beta(x) = e^{-t|\beta|} l_\beta^\lambda(x), \; P_t^\lambda h_\beta(x) = e^{-t\sqrt{|\beta|}} l_\beta^\lambda(x).$$

For $\alpha > 0$, the Fractional Integral or Riesz potential of order α, I_α^γ and I_α^λ with respect to the Gaussian measure and the probabilistic Gamma measure, are defined as in the classical case, by

$$I_\alpha^\gamma = (-L)^{-\alpha/2} \text{ and } I_\alpha^\lambda = (-\mathfrak{L}_\lambda)^{-\alpha/2}.$$

If $f \in L^1(\gamma_d)$ with $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, it can be proved that[6]

$$I_\alpha^\gamma f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} P_t f(x) dt.$$

Now, following[7] and Bochner subordination formula, it can be proved that

$$I_\alpha^\lambda f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} P_t^\lambda f(x) dt,$$

where $f \in L^1(\mu_{d,\lambda})$ with $\int_{\mathbb{R}^d} f(y)\mu_{d,\lambda}(dy) = 0$.

Observe that if $f(x) = h_\beta(x)$ or $f(x) = l_\beta^\lambda(x)$ and $|\beta| > 0$, we get that

$$I_\alpha^\gamma h_\beta(x) = |\beta|^{-\alpha/2} h_\beta(x) \text{ and } I_\alpha^\lambda l_\beta^\lambda(x) = |\beta|^{-\alpha/2} l_\beta^\lambda(x).$$

In a previous paper[8], the Fractional Derivate for the Gaussian measure and the probabilistic Gamma measure were defined as,

$$D_\alpha^\gamma = (-L)^{\alpha/2} \text{ and } D_\alpha^\lambda = (-\mathfrak{L}_\lambda)^{\alpha/2}$$

and when $0 < \alpha < 1$, we can write for f a polynomial

$$D_\alpha^\gamma f(x) = \frac{1}{c_\alpha} \int_0^\infty t^{-\alpha-1} (P_t f(x) - f(x)) dt,$$

$$D_\alpha^\lambda f(x) = \frac{1}{c_\alpha} \int_0^\infty t^{-\alpha-1} (P_t^\lambda f(x) - f(x)) dt.$$

where

$$c_\alpha = \int_0^\infty u^{-\alpha-1}(e^{-u} - 1) du.$$

Again in[7] and [8], we can observe that if $f(x) = h_\beta(x)$ or $f(x) = l_\beta^\lambda(x)$ with $|\beta| > 0$, then

$$D_\alpha^\gamma h_\beta(x) = |\beta|^{\alpha/2} h_\beta(x) \text{ and } D_\alpha^\lambda l_\beta^\lambda(x) = |\beta|^{\alpha/2} l_\beta^\lambda(x).$$

As an application of these functions D_α^γ and D_α^λ, characterization have been given of the Gaussian Sobolev $L_\alpha^p(\gamma_d)$ spaces and Laguerre Sobolev $L_\alpha^p(\mu_{d,\lambda})$ spaces, for $0 < \alpha < 1$ and $1 < p < \infty$. These spaces have been defined as follows[7,9]. Let us consider the norm

$$\|f\|_{p,\alpha} := \left\|(I - L)^{\alpha/2} f\right\|_{p,\gamma_d}$$

then, the Gaussian Sobolev space of order α, is defined as the completation of the polynomials with respect to the norm $\| \;\|_{p,\alpha}$.

Using the $\mathfrak{L}_\lambda$ operator and $\mu_{d,\lambda}$ measure instead L and γ_d, we define Laguerre Sobolev space in a similar way.

In a previous paper, we got the following result[8]

Theorem 2.1: *Let us consider $0 < \alpha < 1$, and $1 < p < \infty$. Then*

i. *If $\{P_n\}_n$ is a sequence of polynomials such that $\lim_{n\to\infty} P_n = f$ in $L_\alpha^p(\gamma_d)$, then $\lim_{n\to\infty} D_\alpha^\gamma P_n$ exists in $L_\alpha^p(\gamma_d)$ and does not depend on the choice of a sequence $\{P_n\}_n$. If $f \in L_\alpha^p(\gamma_d) \bigcap L_\alpha^r(\gamma_d)$, then the limit does not depend on the choice of p or r. Thus $D_\alpha^\gamma f = \lim_{n\to\infty} D_\alpha^\gamma P_n$ in $L_\alpha^p(\gamma_d)$, $f = \lim_{n\to\infty} P_n$ in $L_\alpha^p(\gamma_d)$, $f \in L_\alpha^p(\gamma_d)$ is well defined.*

ii. *$f \in L_\alpha^p(\gamma_d)$ if and only if $D_\alpha^\gamma f \in L^p(\gamma_d)$. Moreover,*

$$B_{p,\alpha} \|f\|_{p,\alpha} \le \left\|D_\alpha^\gamma f\right\|_{p,\gamma_d} \le A_{p,\alpha} \|f\|_{p,\alpha}.$$

Remark 1: For D_α^λ operator with Laguerre polynomials expansions and $\mu_{d,\lambda}$ measure, results similar to the Theorem (2.1) were obtained[7].

RESULTS

We star considering Hermite expansions and γ_d measure. Similar results are obtained to Laguerre expansions and $\mu_{d,\lambda}$ measure.

We define the operators Q_t and Q_t^λ if $\lambda \in (-1,\infty)^d$

$$Q_t f(x) = -t\partial_t P_t f(x), \; Q_t^\lambda f(x) = -t\partial_t P_t^\lambda f(x)$$

and let us consider $\alpha \ge 0$ and $0 < p,q \le \infty$. Following[4], the Gaussian Triebel-Lizorkin $\dot{F}_p^{\alpha,q}(\gamma_d)$ space, is the set of functions for which

$$\text{(3.1)} \quad \|f\|_{\dot{F}_p^{\alpha,q}} = \left\|\left(\int_0^\infty t^{-2\alpha} |Q_t f|^q \frac{dt}{t}\right)^{1/q}\right\|_{p,\gamma_d} < \infty$$

and the Gaussian Besov spaces $\dot{B}_p^{\alpha,q}(\gamma_d)$ can be defined as the set of the functions such that

$$\|f\|_{\dot{B}^{\alpha,q}_{p}} = \left(\int_0^\infty t^{-2\alpha} \|Q_t f\|^q_{p,\gamma_d} \frac{dt}{t} \right)^{1/q} < \infty.$$

Now following[11], we introduced an associated function to the Fractional Derivate defined by

$$(3.2) \quad \mathfrak{D}^{\gamma}_{\alpha} f(x) = \left(\int_0^\infty t^{-2\alpha+1} |\partial_t P_t f(x)|^2 dt \right)^{1/2},$$

where $0 < \alpha < 1$. We can see that $\mathfrak{D}^{\gamma}_{\alpha}$ can be rewritten as

$$(3.3) \quad \mathfrak{D}^{\gamma}_{\alpha} f(x) = \left(\int_0^\infty t^{-2\alpha} |Q_t f(x)|^2 dt \right)^{1/2}.$$

We are going to study the $\mathfrak{D}^{\gamma}_{\alpha}$ operator. First, this operator satisfies $\mathfrak{D}^{\gamma}_{\alpha} h_\beta = 2^{2(\alpha-1)} c_\alpha |\beta|^{\alpha/2} |h_\beta|$, where, $c_\alpha = \int_0^\infty u^{-2\alpha+1} e^{-u} du < \infty$, because $\alpha \in (0,1)$.

In order to show this fact, we consider the change of variable $u = 2\sqrt{|\beta|}t$ and the definition of c_α. Then,

$$\begin{aligned} \mathfrak{D}^{\gamma}_{\alpha} h_\beta(x) &= |\beta| |h_\beta(x)| \left\{ \int_0^\infty t^{-2\alpha+1} e^{-2\sqrt{|\beta|}t} dt \right\}^{1/2} \\ &= 2^{2(\alpha-1)} |\beta|^{\alpha/2} |h_\beta(x)| \left\{ \int_0^\infty u^{-2\alpha+1} e^{-u} du \right\}^{1/2} \\ &= 2^{2(\alpha-1)} c_\alpha |\beta|^{\alpha/2} |h_\beta(x)| \end{aligned}$$

This way if $f = h_\beta$, then $\mathfrak{D}^{\gamma}_{\alpha} f \in L^p(\gamma_d)$ with $1 < p < \infty$. Also, $\mathfrak{D}^{\gamma}_{\alpha}$ is a sub lineal operator. It is easy to see that, $\mathfrak{D}^{\gamma}_{\alpha}(\lambda f) = |\lambda| \mathfrak{D}^{\gamma}_{\alpha}(f)$ $\forall \lambda \in \mathbb{R}$, and considering the space,

$$H = \left\{ u(.,t) : \int_0^\infty |u(.,t)|^2 t^{-2\alpha+1} dt < \infty \right\}$$

with the norm $\|u\|_H = \left(\int_0^\infty |u(.,t)|^2 t^{-2\alpha+1} dt \right)^{1/2}$, we denote $f(x,t) = \partial_t P_t f(x)$ and $g(x,t) = \partial_t P_t g(x)$, then

$$\mathfrak{D}^{\gamma}_{\alpha} f(x) = \|f(x,.)\|_H \text{ and } \mathfrak{D}^{\gamma}_{\alpha} g(x) = \|g(x,.)\|_H,$$

and Minkowski's inequality implies,

$$\mathfrak{D}^{\gamma}_{\alpha}(f+g)(x) \le \mathfrak{D}^{\gamma}_{\alpha} f(x) + \mathfrak{D}^{\gamma}_{\alpha} g(x) \quad \forall x \in \mathbb{R}^d.$$

Therefore, if f is a polynomial then $\mathfrak{D}^{\gamma}_{\alpha} f \in L^p(\gamma_d)$ with $1 < p < \infty$.

Remark 2: For $\lambda \in (-1,\infty)^d$, the Laguerre Triebel Lizorkin spaces $\dot{F}^{\alpha,q}_{p}(\mu_{d,\lambda})$ and the Laguerre Besov spaces $\dot{B}^{\alpha,q}_{p}(\mu_{d,\lambda})$ are defined in a similar way, by using the operator Q^λ_t and $\mu_{d,\lambda}$ measure instead of Q_t and γ_d measure.

Also, we define

$$\mathfrak{D}^{\lambda}_{\alpha} f(x) = \left(\int_0^\infty t^{-2\alpha+1} \left| \frac{\partial}{\partial t} P^\lambda_t f(x) \right|^2 dt \right)^{1/2},$$

for $0 < \alpha < 1$. Immediately we have,

$$\mathfrak{D}^{\lambda}_{\alpha} l^{\lambda}_{\beta} = 2^{2(\alpha-1)} c_\alpha |\beta|^{\alpha/2} |l^{\lambda}_{\beta}|$$

and it is a straightforward exercise to see that $\mathfrak{D}^{\lambda}_{\alpha}$ satisfies the same before properties.

However, we can go further.

Proposition 3.1: *Suppose $f \in C^2_B(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$. Then $\mathfrak{D}^{\gamma}_{\alpha} f \in L^p(\gamma_d)$ for each $1 < p < \infty$.*

Proof: From the Lemma 2.1[8], we can see that

$$|\partial_t P_t f(x)|^2 \le C_{d,f} (d + |x|)^2 e^{-2t}.$$

Using that, $\alpha \in (0,1)$ and (3.2) we get that

$$\mathfrak{D}^{\gamma}_{\alpha} f(x) \le C_{d,f} (d+|x|) \left(\int_0^\infty e^{-2t} t^{-2\alpha+1} dt \right)^{1/2} = C_{d,f,\alpha} (d + |x|).$$

This way, we get that $\|\mathfrak{D}^{\gamma}_{\alpha} f\|^p_{p,\gamma_d} < \infty$. □

Now, we get the following result[12].

Theorem 3.1: *Let $0 < \alpha < 1$ and $f \in L^2_\alpha(\gamma_d)$. Then $\mathfrak{D}^{\gamma}_{\alpha} f \in L^2(\gamma_d)$ and $\|\mathfrak{D}^{\gamma}_{\alpha} f\|_{2,\gamma_d} \le C_\alpha \|f\|_{2,\alpha}$.*

Proof: Let f be a polynomial and considering $\psi = (I-L)^{\alpha/2} f$, which is also a polynomial. Then $J_n \psi = (1+n)^{\alpha/2} J_n f$, and f can be written as

$$f = \sum_{n\ge 0} \left(\frac{1}{1+n} \right)^{\alpha/2} J_n \psi.$$

Trivially, $f, \psi \in L^2(\gamma_d)$. Now, by Parserval identity, we have

$$\int_{\mathbb{R}^d} |\partial_t P_t f(x)|^2 \gamma_d(dx) = \sum_{n\ge 0} \frac{n e^{-2\sqrt{n}t}}{(1+n)^\alpha} \|J_n \psi\|^2_{2,\gamma_d},$$

and by Tonelli's Theorem we obtain,

$$\begin{aligned} \left\| \left(\int_0^\infty t^{-2\alpha+1} |\partial_t P_t f|^2 dt \right)^{1/2} \right\|^2_{2,\gamma_d} &= C_\alpha \sum_{n\ge 0} \left(\frac{n}{1+n} \right)^\alpha \|J_n \psi\|^2_{2,\gamma_d} \\ &= C_\alpha \sum_{n\ge 0} n^\alpha \|J_n f\|^2_{2,\gamma_d} \\ &\le C_\alpha \sum_{n\ge 0} (1+n)^\alpha \|J_n f\|^2_{2,\gamma_d} = C_\alpha \|f\|^2_{2,\alpha}. \ \square \end{aligned}$$

In consequence, we obtain that $\mathfrak{D}^{\gamma}_{\alpha} f \in L^2(\gamma_d)$, when $f \in L^2_\alpha(\gamma_d)$ and $\mathfrak{D}^{\gamma}_{\alpha} f = \sum_{n\ge 0} J_n(\mathfrak{D}^{\gamma}_{\alpha} f)$.

But,

$$\|\mathfrak{D}^{\gamma}_{\alpha} f\|^2_{2,\gamma_d} = C_\alpha \sum_{n\ge 0} n^\alpha \|J_n f\|^2_{2,\gamma_d}$$

and this is equivalent to $\lim_{k\to\infty} \|\mathfrak{D}^{\gamma}_{\alpha} f - S^\alpha_k f\|^2_{2,\gamma_d} = 0$, where

$S_k^\alpha f = \sqrt{C_\alpha}\sum_{n=0}^{k} n^{\alpha/2} J_n f$. Since the Hermite polynomials $\{h_\beta\}_{\beta\in\mathbb{N}^d}$ system is complete, then $\mathfrak{D}_\alpha^\gamma f$ can be written as $\mathfrak{D}_\alpha^\gamma f = \sqrt{C_\alpha}\sum_{n\geq 0} n^{\alpha/2} J_n f$, when $f\in L_\alpha^2(\gamma_d)$ and specially, when f is a polynomial function.

This way, we are ready to show the next result. We got a similar result to the Fractional Derivate operator[10].

Theorem 3.2: *Let us consider* $0<\alpha<1$ *and* $1<p<\infty$. *Then*

i. *If* $\{P_n\}_n$ *is a sequence of polynomials such that* $\lim_{n\to\infty} P_n = f$ *in* $L_\alpha^p(\gamma_d)$, *then* $\lim_{n\to\infty}\mathfrak{D}_\alpha^\gamma P_n$ *exists in* $L_\alpha^p(\gamma_d)$ *and does not depend on the choice of a sequence.* $\{P_n\}_n$ *If* $f\in L_\alpha^p(\gamma_d)\cap L_\alpha^r(\gamma_d)$, *then the limit does not depend on the choice of p or r. Thus* $\mathfrak{D}_\alpha^\gamma f = \lim_{n\to\infty}\mathfrak{D}_\alpha^\gamma P_n$ in $L_\alpha^p(\gamma_d)$, $f=\lim_{n\to\infty} P_n$ in $L_\alpha^p(\gamma_d)$, $f\in L_\alpha^p(\gamma_d)$ *is well defined.*

ii. $f\in L_\alpha^p(\gamma_d)$ *if and only if* $\mathfrak{D}_\alpha^\gamma f\in L^p(\gamma_d)$. *Moreover,*

$$(3.4)\quad B_{p,\alpha}\|f\|_{p,\alpha} \leq \|\mathfrak{D}_\alpha^\gamma f\|_{p,\gamma_d} \leq A_{p,\alpha}\|f\|_{p,\alpha}.$$

Proof: Let f be a polynomial and considering $\psi=(I-L)^{\alpha/2} f$, which is also a polynomial, then $\mathfrak{D}_\alpha^\gamma f$ can be written as

$$\mathfrak{D}_\alpha^\gamma f = \sqrt{C_\alpha}\sum_{n\geq 0}\left(\frac{n}{1+n}\right)^{\alpha/2} J_n\psi,$$

since $\|f\|_{p,\alpha} = \|\psi\|_{p,\gamma_d}$.

By Meyer's Multipliers Theorem, using the multiplier $h(z)=(z+1)^{-\alpha/2}$, we obtain that

$$\|\mathfrak{D}_\alpha^\gamma f\|_{p,\gamma_d} \leq A_{p,\alpha}\|\psi\|_{p,\gamma_d} = A_{p,\alpha}\|f\|_{p,\alpha}.$$

To prove the converse, suppose f polynomial, then $\mathfrak{D}_\alpha^\gamma f\in L^p(\gamma_d)$. Consider

$$\psi=(I-L)^{\alpha/2} f = \sum_{n\geq 0}(1+n)^{\alpha/2} J_n f = \sum_{n\geq 0}\left(\frac{1+n}{n}\right)^{\alpha/2} J_n\left(\mathfrak{D}_\alpha^\gamma f\right).$$

By Meyer's Multipliers Theorem, using the multiplier $h(z)=(z+1)^{\alpha/2}$, we have

$$\|f\|_{p,\gamma_d} = \|\psi\|_{p,\alpha} \leq B_{p,\alpha}\|\mathfrak{D}_\alpha^\gamma f\|_{p,\gamma_d}.$$

Thus we get (3.4) for polynomials. Then we can prove part i) using (3.4), the completeness of $L_\alpha^p(\gamma_d)$ and the fact that for each $1<p<\infty$ we have,

$$L_\beta^p(\gamma_d)\subset L_\alpha^p(\gamma_d),\ \text{when}\ 0\leq\alpha<\beta.$$

Finally we get (3.4) for any $f\in L_\alpha^p(\gamma_d)$. □

Remark 3: In the Laguerre case we have,

Proposition 3.2: *Suppose* $f\in C_B^2(\mathbb{R}^d)$, *such that* $\int_{\mathbb{R}^d} f(y)\mu_{d,\lambda}(dy)=0$. *Then* $\mathfrak{D}_\alpha^\lambda f\in L^p(\mu_{d,\lambda})$ *for each* $1<p<\infty$ *and* $\lambda\in(-1,\infty)^d$.

Proof: This fact follows from[7]

$$\left|\partial_t M_t^\lambda f(x)\right| \leq C_{f,d,\lambda}(1+|x|)e^{-t}$$

and using Bochner' subordination formula, then we use the same argument to the Proposition 3.1. □

By other hand, if you consider Laguerre expansions instead of Hermite expansions and Meyer's Multipliers Theorem for Laguerre version[7], immediately we obtain the Theorem 3.1 and the Theorem 3.2.

Following[1], let us now consider three spaces of functions. These spaces are particular cases of Gausssian Triebel Lizorkin spaces and Laguerre Triebel Lizorkin spaces.

a) The spaces $L^p(\gamma_d)$ ***and*** $L^p(\mu_{d,\lambda})$

Observe that if $\alpha=0$, then (3.2) is a Littlewood Paley *g*-function defined[13] and by Littlewood Paley theory it is know that

$$\|f\|_{p,\gamma_d} \approx \left\|\left(\int_0^\infty t\left|\partial_t P_t f\right|^2 dt\right)^{1/2}\right\|_{p,\gamma_d}.$$

Then, from the definition of the Gaussian Triebel Lizorkin spaces, we have

$$L^p(\gamma_d)\sim \dot{F}_p^{0,2}(\gamma_d)\ \text{when}\ 1<p<\infty.$$

But, when you consider $\lambda_i = \frac{n_i}{2}-1$, $n_i\in\mathbb{N}$ for all $i=1\ldots d$, it has been showed that[14]

$$\|f\|_{p,\mu_{d,\lambda}} \approx \left\|\left(\int_0^\infty t\left|\partial_t P_t^\lambda f\right|^2 dt\right)^{1/2}\right\|_{p,\mu_{d,\lambda}}$$

Then immediately we get

$$L^p(\mu_{d,\lambda})\sim \dot{F}_p^{0,2}(\mu_{d,\lambda}).$$

b) Gaussian Sobolev spaces $L_\alpha^p(\gamma_d)$ ***and Laguerre spaces*** $L_\alpha^p(\mu_{d,\lambda})$

Suppose $1<p<\infty$, $\alpha\in(0,1)$ and $\lambda\in(-1,\infty)^d$. First, let us consider the Gaussian Triebel Lizorkin $\dot{F}_p^{\alpha,2}(\gamma_d)$ space. We define modulo constants,

$$\|f\|_{\dot{F}_p^{\alpha,2}} := \|D_\alpha^\gamma f\|_{p,\gamma_d}.$$

Then, the Gaussian Triebel-Lizorkin space $\dot{F}_p^{\alpha,2}(\gamma_d)$ is defined as the completion of the polynomials with respect to the norm $\|\ \|_{\dot{F}_p^{\alpha,2}}$. Using the Fractional Derivate and the Theorem 2.1[8], we can see that

$$L^p(\gamma_d)\sim \dot{F}_p^{\alpha,2}(\gamma_d)\ \text{for}\ 1<p<\infty\ \text{and}\ 0<\alpha<1.$$

But now, we get an equivalent way to define Gaussian

Triebel Lizorkin $\dot{F}_p^{\alpha,2}(\gamma_d)$ space. Using (3.1) and (3.3), it is immediately that

$$\|f\|_{\dot{F}_p^{\alpha,2}} := \left\|\left(\int_0^\infty t^{-2\alpha}|Q_t f|^2\,dt\right)^{1/2}\right\|_{p,\gamma_d} = \|\mathfrak{D}_\alpha^\gamma f\|_{p,\gamma_d}.$$

Then, from the Theorem 3.2, we obtain the following corollary

Corollary 3.1: *Let $1 < p < \infty$ and $\alpha \in (0,1)$ $f \in \dot{F}_p^{\alpha,2}(\gamma_d)$ if and only if $f \in L_\alpha^p(\gamma_d)$. Moreover,*

$$B_{p,\alpha}\|f\|_{p,\alpha} \le \|f\|_{\dot{F}_p^{\alpha,2}} \le A_{p,\alpha}\|f\|_{p,\alpha}.$$

Therefore, using the $\mathfrak{D}_\alpha^\gamma$ operator we have again,

$$L^p(\gamma_d) \sim \dot{F}_p^{\alpha,2}(\gamma_d) \text{ for } 1 < p < \infty \text{ and } 0 < \alpha < 1.$$

In a similar argument by means the Laguerre Fractional Derivate D_α^λ we obtain that

$$L^p(\mu_{d,\lambda}) \sim \dot{F}_p^{\alpha,2}(\mu_{d,\lambda}) \text{ for } 1 < p < \infty \text{ and } 0 < \alpha < 1,$$

where, $\|f\|_{\dot{F}_p^{\alpha,2}} := \|D_\alpha^\lambda f\|_{p,\mu_{d,\lambda}}$.

Now, using the definition of $\dot{F}_p^{\alpha,2}(\gamma_d)$ space, $\mathfrak{D}_\alpha^\lambda$ operator, we get the same result.

e) Carleson measures with respect to γ_d and $\mu_{d,\lambda}$

Following[15] we are going to study the space *C*, whose elements are measures on $\mathbb{R}_+^d$.

If $B = B(x,r)$ is an open ball in $\mathbb{R}^d$, then its tent *T(B)* is the closed set in $\mathbb{R}_+^d$, defined by

$$T(B) = \{(y,t) : |y-x| \le r-t\}.$$

Given a Borel measure $d\mu$ on $\mathbb{R}_+^d$, we define the function $C(d\mu)$, like in the classic case, by

$$C(d\mu)\,(x) = \sup_{\{B/x\in B\}} \frac{1}{\gamma_d(B)}\int_{T(B)}|d\mu|.$$

Then, we define *C*, to be the space of measures $d\mu$ for which $C(d\mu)$ is a bounded function and set

$$\|d\mu\|_C = \sup_{x\in\mathbb{R}^d}|C(d\mu)(x)|. \quad (3.5)$$

Each $d\mu$ is called a Carleson measure, and (3.5) is the Carleson norm of $d\mu$.

Now, we consider the Gaussian Triebel Lizorkin space $\dot{F}_\infty^{0,2}(\gamma_d)$, where

$$\|f\|_{\dot{F}_\infty^{0,2}} = \sup_{x\in\mathbb{R}^d}\left(\int_0^\infty |Q_t f(x)|^2\frac{dt}{t}\right)^{1/2}.$$

Then, when we consider

$$d\mu(x,t) = t|\partial_t P_t f(x)|^2\,\gamma_d(dx)dt,$$

we obtain the following version of Fefferman's Theorem for the Gaussian measure.

Theorem 3.5: *Suppose $f \in \dot{F}_\infty^{0,2}(\gamma_d)$ and*

$$d\mu(x,t) = t|\partial_t P_t f(x)|^2\,\gamma_d(dx)dt.$$

Then $d\mu$ is a Carleson measure, with

$$\|d\mu\|_C \le \|f\|_{\dot{F}_\infty^{0,2}}^2.$$

Proof: Let us consider $B \subset \mathbb{R}^d$ with radio $r>0$, fix and arbitrary. Then we can see that

$$\frac{1}{\gamma_d(B)}\iint_{T(B)}|t\partial_t P_t f(x)|^2\,\gamma_d(dx)\frac{dt}{t} \le \frac{1}{\gamma_d(B)}\int_B\int_0^r |t\partial_t P_t f(x)|^2\,\gamma_d(dx)\frac{dt}{t}$$

$$\le \frac{1}{\gamma_d(B)}\int_B\int_0^\infty |t\partial_t P_t f(x)|^2\,\gamma_d(dx)\frac{dt}{t} \le \|f\|_{\dot{F}_\infty^{0,2}}^2.$$

Then

$$\frac{1}{\gamma_d(B)}\iint_{T(B)}|t\partial_t P_t f(x)|^2\,\gamma_d(dx)\frac{dt}{t} \le \|f\|_{\dot{F}_\infty^{0,2}}^2,$$

and we get the theorem. □

Now we give the following version of Carleson's problem

Theorem 3.6: *Suppose $g \in \dot{F}_\infty^{0,2}(\gamma_d)$ and let us consider $d\mu(x,t) = t|\partial_t P_t g(x)|^2\,\gamma_d(dx)dt$.*

Then,

$$\int_{\mathbb{R}_+^d}|P_t f(x)|^2\,d\mu(x,t) \le C_2\|g\|_{\dot{F}_\infty^{0,2}}^2\int_{\mathbb{R}^d}|f(x)|^2\,\gamma_d(dx).$$

$\forall f \in L^2(\gamma_d)$.

Proof: Using the definition of $d\mu$ we get,

$$\int_{\mathbb{R}_+^d}|P_t f(x)|^2\,d\mu(x,t) \le \int_{\mathbb{R}^d}\left(P_t^+ f(x)\right)^2\int_0^\infty t|\partial_t P_t g(x)|^2\,dt\gamma_d(dx),$$

where

$$P^+ f(x) = \sup_{t>0}|P_t f(x)|,$$

and using the definiton of $\|g\|_{\dot{F}_\infty^{0,2}}$ and the fact that[10] $\|P^+ f\|_{p,\gamma_d} \le C_p\|f\|_{p,\gamma_d}$ $1 < p < \infty$, we obtain the result of the theorem. □

Now we consider

$$\Gamma_\gamma(x) = \left\{(y,t) \in \mathbb{R}_+^{d+1} : |x-y| < \left(t \wedge |x|^{-1} \wedge 1\right)\right\},$$

is what is called a Gaussian cone of aperture 1 and vertex $x \in \mathbb{R}^d$ [16] and the nontangential maximal function $P^* f(x) = \sup_{(y,t)\in\Gamma_\gamma(x)}|P_t f(x)|$.

Following[15], let *N* be the linear space of all (everywhere defined) functions such that $P^* f \in L^1(\gamma_d)$, with the norm $\|f\|_N := \|P^* f\|_{1,\gamma_d}$.

For example, if $f \in C_B^2(\mathbb{R}^d)$, from the Lemma 2.1[8], we can see again that, $|\partial_t P_t f(y)|^2 \le C_{d,f}(d+|y|)e^{-t}$ and consequently, from the definition of P^*,

$$P^* f(x) \le C_{d,f}(1+d+|x|).$$

This way, $f \in N$. In this context, we get the following result[15]

Theorem 3.7: *Suppose* $f \in \dot{F}_{\infty}^{0,2}(\gamma_d)$ *and let us consider* $d\mu(x,t) = t|\partial_t P_t f(x)|^2 \gamma_d(dx)dt$.

If $f \in N$. *we have,* $\left|\int_{\mathbb{R}_+^d} P_t h(x) d\mu(x,t)\right| \le c\|d\mu\|_C \|h\|_N$

Proof: The proof of the theorem is similar to the classical case, where the Lebesgue measure it was considered. This proof is based on two simple but key observations. First

(3.6) $\quad \{(x,t) : |P_t h(x)| > \alpha\} \subset T(O),$

where $O = \{x : P^* h(x) > \alpha\}$,and

$$T(O) = \bigcup_{x \in O} T(B(x, dist(x, O^c)))\cdot$$

Since, if (x,t) is such that $|P_t h(x)| > \alpha$, then for any $y \in B(x,r)$ with $r \le (t \wedge |x|^{-1} \wedge 1)$, we have that $P^* h(x) > \alpha$. But

$$\underset{\{(y,t): |x-y|<t\}}{Sup} |P_t f(x)| \ge P^* f(y) > \alpha$$

thus, $B = B(x,t) \subset O$ and $(x,t) \in T(B)$.

The second key observation is as follows. If we assume that $\|d\mu\|_C \le 1$, then this implies that

(3.7) $\quad \mu(T(O)) \le c\gamma_d(O),$

for any open set $O \subset \mathbb{R}^d$. This true in our case, because the principal key ingredient to show (3.7) is the Whitney decomposition[12]. This decomposition does not involve measure theory, but deals with the geometric structure of general closed set O^c in $\mathbb{R}^d$. This way, we can repeat the same argument like in the classical case.

Now, using (3.6) and (3.7) we have,

$$\mu(\{(x,t) : |P_t h(x)| > \alpha\}) \le \mu(T(O)) \le c\|d\mu\|_C \gamma_d(O)$$

and integrating both sides with respect α we obtain,

$$\int_{\mathbb{R}_+^d} |P_t h(x)| d\mu(x,t) \le c\|d\mu\|_C \int_{\mathbb{R}^d} P^* h(x) \gamma_d(dx). \ \square$$

This theorem implies the following corollary. This corollary is a general version from Carleson's problem.

Corollary 3.2: *Suppose* $f \in \dot{F}_{\infty}^{0,2}(\gamma_d)$ *and let us consider* $d\mu(x,t) = t|\partial_t P_t f(x)|^2 \gamma_d(dx)dt$.

If $h \in L^p(\gamma_d)$, *for some* $1 < p < \infty$ *then we have,*

$$\int_{\mathbb{R}_+^d} |P_t h(x)|^p d\mu(x,t) \le C_p \|d\mu\|_C \|h\|_{p,\gamma_d}.$$

Proof: In fact, the corollary follows from the Theorem (3.7), by replacing $|P_t h(x)|$ by $|P_t h(x)|^p$ and the $L^p(\gamma_d)$ continuity for $1 < p < \infty$ of the maximal operator $P^* h$ [16] $\square$

Remark 4: Now, considering $\mu_{d,\lambda}$ and the function $C(d\mu)$, defined by

$$C(d\mu)(x) = \underset{\{B/x \in B\}}{Sup} \frac{1}{\mu_{d,\lambda}(B)} \int_{T(B)} |d\mu|$$

similar to before case, we say that $d\mu$ is a Carleson measure if $C(d\mu)$ is a bounded function.

Also, we consider the Laguerre Triebel Lizorkin space $\dot{F}_{\infty}^{0,2}(\mu_{d,\lambda})$, where

$$\|f\|_{\dot{F}_{\infty}^{0,2}} = \underset{x \in \mathbb{R}^d}{Sup}\left(\int_0^{\infty} |Q_t^{\lambda} f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

and we get the Theorem 3.3 and the Theorem 3.4, taking P_t^{λ} and $\mu_{d,\lambda}$ instead of P_t and γ_d respectively and considering the maximal function

$$P_{\lambda}^* f(x) = \underset{t>0}{Sup}|P_t^{\lambda} f(x)|,$$

which it is $L^p(\mu_{d,\lambda})$ -continuous when $1 < p < \infty$ [10].

To get the others results, it is necessary to find appropriate definitions of the Laguerre cone and P_{λ}^* maximal function for Laguerre semigroup.

The last result in this article, is the following relation between Gaussian Triebel-Lizorkin spaces and Gaussian Besov spaces.

Proposition 3.3: *Let* $\alpha > 0$, $p > 0$, $q > 0$, *and* $r > 0$, *such that*

i. If $q > p$ *then* $\dot{F}_p^{\alpha,q}(\gamma_d) \subset \dot{B}_p^{\alpha,q}(\gamma_d)$.

ii. If $p > q$ *then* $\dot{B}_p^{\alpha,q}(\gamma_d) \subset \dot{F}_p^{\alpha,q}(\gamma_d)$.

iii. If $r > q$ *then* $\dot{B}_p^{\alpha,q}(\gamma_d) \subset \dot{B}_p^{\alpha,r}(\gamma_d)$.

Proof

i. Suppose $q > p$. By using Minkowski's integral inequality we have that

$$\|f\|_{\dot{B}_p^{\alpha,q}}^p = \left(\int_0^{\infty} t^{-2\alpha} \|Q_t f\|_{p,\gamma_d}^q \frac{dt}{t}\right)^{p/q}$$

$$= \left(\int_0^{\infty} t^{-2\alpha}\left(\int_{\mathbb{R}^d} |Q_t f(x)|^p \gamma_d(dx)\right)^{q/p} \frac{dt}{t}\right)^{p/q}$$

$$\le \int_{\mathbb{R}^d}\left(\int_0^{\infty} t^{-2\alpha} |Q_t f(x)|^q \frac{dt}{t}\right)^{p/q} \gamma_d(dx) = \|f\|_{\dot{F}_p^{\alpha,q}}^p.$$

ii. Analogously, by using Minkowski's integral inequality again, we have

$$\|f\|_{\dot{F}_p^{\alpha,q}}^q = \left(\int_{\mathbb{R}^d}\left(\int_0^{\infty} t^{-2\alpha} |Q_t f(x)|^q \frac{dt}{t}\right)^{p/q} \gamma_d(dx)\right)^{q/p}$$

$$\le \left(\int_0^{\infty} t^{-2\alpha} \|Q_t f\|_{p,\gamma_d}^q \frac{dt}{t}\right)^{q/p} = \|f\|_{\dot{B}_p^{\alpha,q}}^q.$$

iii. If $f \in \dot{B}_p^{\alpha,q}(\gamma_d)$, we can suppose that $\|f\|_{\dot{B}_p^{\alpha,q}}^q \le 1$.

This way we have that $\|Q_t f\|_{p,\gamma_d} \leq 1$, and then $\|Q_t f\|^r_{p,\gamma_d} \leq \|Q_t f\|^q_{p,\gamma_d}$ when $q \leq r$. Consequently $f \in \dot{B}^{\alpha,r}_p(\gamma_d)$.

Remark 5: Finally, it is very easy to get a similar version of the Proposition 3.3 with respect to Laguerre semigroup and $\mu_{d,\lambda}$ measure.

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