Global Behavior of a Higher Order Difference Equation

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Abstract: In this work we study the asymptotic stability of the nonnegative equilibrium points of the difference equation $x_{n+1} = \dfrac{Ax_{n-1}}{B+C\prod\limits_{i=l}^{k} x_{n-2i}}, n = 0,1,\ldots$ where A, B, C are nonnegative real numbers and l, k are nonnegative integers, $l \le k$. We discuss the conditions under which there exist prime period two solutions and semicycles. Finally we investigate the oscillation and the existence of unbounded solutions.

Key words: Difference equation, periodic solution, globally asymptotically stable

INTRODUCTION

Difference equations have always played an important role in the construction and analysis of mathematical models of biology, ecology, physics, economic processes, etc[1].

The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. Cinar[2] examined the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}, n = 0,1,..$$

Where a and b are nonnegative real numbers.
Xiaofan yang *et al.*[3] investigated the asymptotic behavior of solutions of the difference equations

$$x_{n+1} = \frac{ax_n + bx_{n-1}}{c+dx_n x_{n-1}}, n = 0,1,\ldots$$

Where $a \ge 0$ and b,c,d are nonnegative real numbers.
Gibbons *et al.*[4] investigated the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}, n = 0,1,\ldots \qquad (1.1)$$

where $\beta > 0$ and $\alpha, \gamma \ge 0$.
In this study, we study the global asymptotic stability of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B+C\prod\limits_{i=l}^{k} x_{n-2i}}, n = 0,1,\ldots \qquad (1.2)$$

where A,B,C are nonnegative real numbers and l,k are nonnegative integers, $l \le k$.

The following particular cases can be obtained:

1. When $A = 0$, equation (1.2) reduces to the equation $x_{n+1} = 0, n = 0,1,\ldots$

2. When $B = 0$, equation (1.2) reduces to the equation $x_{n+1} = \dfrac{Ax_{n-1}}{C\prod\limits_{i=l}^{k} x_{n-2i}}, n = 0,1,\cdots$, this equation can be reduced to the linear difference equation

$$y_{n+1} - y_{n-1} + y_{n-2l} + \cdots + y_{n-2k} = \gamma, n = 0,1,\ldots$$

by taking $y_n = x_n, \gamma = \ln \frac{A}{C}$

3. When $C = 0$, equation (1.2) reduces to the equation $x_{n+1} = \frac{A}{B} x_{n-1}, n = 0,1,\ldots$ which is a linear difference equation.

4. When $l = k = 0$, and $C = 1$ equation (1.2) yields equation (1.1) with $\alpha = 0$. For various values of l and k, we can get more equations.

Preliminaries: Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1},\ldots,x_{n-k}), n = 0,1,\ldots \qquad (2.1)$$

Where $f : R^{k+1} \to R$

Definition 2.1[5]**:** An equilibrium point for equation (2.1) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x},\bar{x},\ldots\bar{x})$.

Definition 2.2 [5]

1. An equilibrium point $\bar{x}$ for equation (2.1) is called locally stable if for every $\varepsilon > 0$, $\exists \delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1},\ldots,x_0 \in]\bar{x} - \delta, \bar{x} - \delta[$ is such

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that $x_n \in]\bar{x} - \varepsilon, \bar{x} + \varepsilon[\, \forall n \in N$. Otherwise $\bar{x}$ is said to be unstable.

2. The equilibrium point $\bar{x}$ of equation (2.1) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in]\bar{x} - \gamma, \bar{x} - \gamma[$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

3. An equilibrium point $\bar{x}$ for equation (2.1) is called global attractor if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \to \infty$.

4. The equilibrium point $\bar{x}$ for equation (2.1) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with equation (2.1) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \bar{x}, ..., \bar{x}) y_{n-i}, n = 0, 1, ... \qquad (2.2)$$

the characteristic equation associated with equation (2.2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \bar{x}, ..., \bar{x}) \lambda^{k-i} = 0 \qquad (2.3)$$

Theorem 2.3[5]: Assume that f is a C^1 function and let $\bar{x}$ be an equilibrium point of equation (2.2). Then the following statements are true:

1. If all roots of equation (2.3) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.

2. If at least one root of equation (2.3) has absolute value greater than one, then $\bar{x}$ is unstable.

The change of variables $x_n = {}^{k-l+1}\sqrt{\frac{B}{C}} y_n$ reduces equation (1.2) to the difference equation

$$y_{n+1} = \frac{\gamma y_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}}, n = 0, 1, ... \qquad (2.4)$$

where $\gamma = \frac{A}{B}$.

Linearized stability analysis: In this section we study the asymptotic stability of the nonnegative equilibrium points of equation (2.4). We can see that equation (2.4) has two nonnegative equilibrium points $\bar{y} = 0$ and $\bar{y} = {}^{k-l+1}\sqrt{\gamma - 1}$ when $\gamma > 1$ and the zero equilibrium only when $\gamma \le 1$.

The linearized equation associated with equation (2.4) about $\bar{y}$ is

$$z_{n+1} - \frac{\gamma}{1 + \bar{y}^{k-l+1}} z_{n-1} + \frac{\gamma \bar{y}^{k-l}}{(1 + \bar{y}^{k-l+1})^2} \sum_{i=l}^{k} z_{n-2i}$$
$$= 0, n = 0, 1, ... \qquad (3.1)$$

the characteristic equation associated with this equation is

$$\lambda^{2k+1} - \frac{\gamma}{1 + \bar{y}^{k-l+1}} \lambda^{2k-1} +$$

$$\frac{\gamma \bar{y}^{k-l}}{(1 + \bar{y}^{k-l+1})^2} \sum_{i=l}^{k} \lambda^{2k-2i} = 0. \qquad (3.2)$$

We summarize the results of this section in the following theorem.

Theorem 3.1

1. If $\gamma < 1$, then the zero equilibrium point is locally asymptotically stable.

2. If $\gamma > 1$, then the equilibrium points $\bar{y} = 0$ and $\bar{y} = {}^{k-l+1}\sqrt{\gamma - 1}$ are unstable (saddle points).

Proof: The linearized equation associated with equation (2.4) about $\bar{y} = 0$ is

$$z_{n+1} - \gamma z_{n-1} = 0, n = 0, 1, ...$$

the characteristic equation associated with this equation is

$$\lambda^{2k+1} - \gamma \lambda^{2k-1} = 0.$$

So $\lambda = 0, \pm\sqrt{\gamma}$.

1. If $\gamma < 1$, then $|\lambda| < 1$ for all roots and $\bar{y} = 0$ is locally asymptotically stable.

2. If $\gamma > 1$, it follows that $\bar{y} = 0$ is unstable (saddle point).

The linearized equation (3.1) about $\bar{y} = {}^{k-l+1}\sqrt{\gamma - 1}$ becomes

$$z_{n+1} - z_{n-1} + (1 - \frac{1}{\gamma}) \sum_{i=l}^{k} z_{n-2i} = 0, n = 0, 1, ...$$

the associated characteristic equation is

$$\lambda^{2k+1} - \lambda^{2k-1} + (1 - \frac{1}{\gamma}) \sum_{i=l}^{k} \lambda^{2k-2i} = 0.$$

Let $f(\lambda) = \lambda^{2k+1} - \lambda^{2k-1} + (1 - \frac{1}{\gamma}) \sum_{i=l}^{k} \lambda^{2k-2i}$.

We can see that $f(\lambda)$ has a root in $(-\infty, -1)$. Then the point $\bar{y} = {}^{k-l+1}\sqrt{\gamma - 1}$ is unstable (saddle point).

Global behavior of equation (2.4)
Theorem 4.1: If $\gamma < 1$, then the zero equilibrium point is globally asymptotically stable.

Proof: Let $\{y_n\}$ be a solution of equation (2.4). Hence $y_{n+1} = \frac{\gamma y_{n-1}}{1 + \prod_{i=l}^{k} y_{n-2i}} < \gamma y_{n-1}, n = 0, 1, ...$. This implies that $\lim_{n \to \infty} y_n = 0$.

In view of theorem (3.1), $\bar{y}=0$ is globally asymptotically stable.

Existence of prime period two solutions: This section is devoted to discuss the condition under which there exist prime period two solutions.

Theorem 5.1: A necessary and sufficient condition for equation (2.4) to have a prime period two solution is that $\gamma=1$. In this case the prime period two solution is of the form $...,0,\varphi,0,\varphi,0,...$ where $\varphi>0$. Furthermore, every solution converges to a period two solution.

Proof
Sufficiency: Let $\gamma=1$, then for every $\varphi>0$, we have $...,0,\varphi,0,\varphi,0,...$ is a prime period two solution.

Necessity: Assume that equation (2.4) has a prime period two solution $...,\psi,\varphi,\psi,\varphi,\psi,...$ then

$$\varphi=\frac{\gamma\varphi}{1+\psi^{k-l+1}},\psi=\frac{\gamma\psi}{1+\varphi^{k-l+1}}.$$

Hence

$$(\varphi-\psi)+\varphi\psi(\psi^{k-l}-\varphi^{k-l})=\gamma(\varphi-\psi),$$

this implies

$$\varphi\psi(\frac{\psi^{k-l}-\varphi^{k-l}}{\psi-\varphi})=1-\gamma. \qquad (5.1)$$

So $\gamma\leq 1$. Similarly,

$$\varphi\psi\frac{(\psi^{k-l}+\varphi^{k-l})}{(\varphi+\psi)}=\gamma-1. \qquad (5.2)$$

So $\gamma\geq 1$. Therefore $\gamma=1$. From (5.2) we have $\varphi\psi=0$ and the solution is of the form $...,0,\varphi,0,\varphi,0,...$ with $\varphi>0$.

Now let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) with $\gamma=1$.

Then

$$y_{n+1}=\frac{y_{n-1}}{1+\prod_{i=l}^{k}y_{n-2i}}<y_{n-1},n=0,1,...$$

and so the even terms $\{y_{2n}\}_{n=-k}^{\infty}$ decreases to a limit φ and the odd terms $\{y_{2n+1}\}_{n=-k}^{\infty}$ decreases to a limit ψ,

where $\varphi=\frac{\varphi}{1+\psi^{k-l+1}},\psi=\frac{\psi}{1+\varphi^{k-l+1}}$.

Then $\psi^{k-l+1}\varphi=0$ and $\varphi^{k-l+1}\psi=0$. Therefore, $\{y_n\}_{n=-2k}^{\infty}$ converges to the periodic solution $...,0,\varphi,0,\varphi,0,...$ with $\varphi>0$.

Semicycle analysis: Here we discuss the existence of semicycles. We need the following theorem to obtain the main result of this section.

Theorem 6.1: Assume that $f\in C([0,\infty[^{2k+1},[0,\infty[)$ is increasing in the even arguments and decreasing in the others. Let $\bar{y}$ be an equilibrium point for the difference equation

$$y_{n+1}=f(y_n,y_{n-1},...,y_{n-2k}),n=0,1,... \qquad (6.1)$$

Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (6.1) such that either, (C_1) $y_{-2k},y_{-2k+2},...,y_0>\bar{y}$, and $y_{-2k+1},y_{-2k+3},...,y_{-1}<\bar{y}$.
Or
(C_2) $y_{-2k},y_{-2k+2},...,y_0 < \bar{y}$ and $y_{-2k+1},y_{-2k+3},...,y_{-1}>\bar{y}$. is satisfied, then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about $\bar{y}$ with semicycles of length one.

Proof: Assume that f is increasing in the even arguments and decreasing in the others.
Let f be satisfying condition (C_1) we have

$$y_1=f(y_0,y_{-1},...,y_{-2k})<f(\bar{y},y_{-1},\bar{y},...,\bar{y})<$$
$$f(\bar{y},\bar{y},\bar{y},...,\bar{y})=\bar{y},$$
$$y_2=f(y_1,y_0,...,y_{-2k+1})>f(\bar{y},y_0,\bar{y},...,\bar{y})>$$
$$f(\bar{y},\bar{y},\bar{y},...,\bar{y})=\bar{y}.$$

By induction we obtain the result.
If f satisfies condition (C_2), we can prove the result similarly.

Corollary 6.2: Assume that $\gamma>1$ and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) such that either (C_1) or (C_2) is satisfied. Then $\{y_n\}_{n=-2k}^{\infty}$ oscillates about the positive equilibrium point $\bar{y}=\sqrt[k-l+1]{\gamma-1}$ with semicycles of length one.

Proof: The proof follows directly from the previous theorem.

Existence of unbounded solutions: Finally we show that, under certain initial conditions, unbounded solution will be obtained.

Theorem 7.1: Assume that $\gamma>1$. Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) and $\bar{y}=\sqrt[k-l+1]{\gamma-1}$, the positive equilibrium point. Then the following statements are true:

1. If $y_{-2k},y_{-2k+2},...,y_0>\bar{y}$ and $y_{-2k+1},y_{-2k+3},...,y_{-1}<\bar{y}$, then $\{y_{2n}\}_{n=-k}^{\infty}$ increases to ∞ and $\{y_{2n+1}\}_{n=-k}^{\infty}$ decreases to 0.

2. If $y_{-2k}, y_{-2k+2}, ..., y_0 < \overline{y}$ and $y_{-2k+1}, y_{-2k+3}, ..., y_{-1} > \overline{y}$, then $\{y_{2n}\}_{n=-k}^{\infty}$ decreases to 0 and $\{y_{2n+1}\}_{n=-k}^{\infty}$ increases to ∞.

Proof

1. Let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of equation (2.4) with initial conditions, $y_{-2k}, y_{-2k+2}, ..., y_0 > \overline{y}$ and $y_{-2k+1}, y_{-2k+3}, ..., y_{-1} < \overline{y}$. Then

$$y_{2n+2} = \frac{\gamma y_{2n}}{1+\prod_{i=l}^{k} y_{2n-2i+1}} > \frac{\gamma y_{2n}}{1+\overline{y}^{k-l+1}} = y_{2n}$$

and

$$y_{2n+3} = \frac{\gamma y_{2n+1}}{1+\prod_{i=l}^{k} y_{2n-2i+2}} < \frac{\gamma y_{2n+1}}{1+\overline{y}^{k-l+1}} = y_{2n+1}.$$

So $\{y_{2n}\}_{n=-k}^{\infty}$ increases to ∞ and $\{y_{2n+1}\}_{n=-k}^{\infty}$ decreases to 0.

2. The proof is similar and will be omitted.

REFERENCES

1. Mickens, R.E., 1990. Difference Equations, Theory and Applications. 2nd Edn, Van Nostrand Renhold.

2. Cinar, C., 2004. On the positive solution of the difference equation $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$. Appl. Math. Comput., 156: 587-590.

3. Yang, X., W. Su, B. Chen, G. M. Megson and D. J. Evans, 2004. On the recursive sequence $x_{n+1} = \frac{ax_n + bx_{n-1}}{c+dx_n x_{n-1}}$. Appl. Mathe. Comput., doi:10.1016/j.amc.2004.03.023.

4. Gibbons, C.H., M.R.S. Kulenović and G. Ladas, 2000. On the recursive sequence $x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n}$. Math. Sci. Res. Hot-Line, 4: 1-11.

5. Kocic, V.L. and G. Ladas, 1993. Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic, Dordrecht.