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Complex Specializations of Krammer's Representation of the Braid Group, B_3

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Abstract: Problem statement: Classifying irreducible complex representations of an abstract group has been always a problem of interest in the field of group representations. In our study, we considered a linear representation of the braid group on three strings, namely, Krammer's representation. The objective of our work was to study the irreducibility of a specialization of Krammer's representation. **Approach:** We specialized the indeterminates used in defining the representation to non zero complex numbers and worked on finding invariant subspaces under certain conditions on the indeterminates. **Results:** we found a necessary and sufficient condition that guarantees the irreducibility of Krammer's representation of the braid group on three strings. **Conclusion:** This was a logical extension to previous results concerning the irreducibility of complex specializations of the Burau representation. The next step is to generalize our result for any n, which might enable us to characterize all irreducible Krammer's representations of various degrees.

Key words: Braid group, magnus representation

INTRODUCTION

Let B_n be the braid group on n strings. There are many kinds of representations of B_n. The earliest was the Artin representation, which is an embedding $B_n \to Aut(F_n)$, the automorphism group of a free group on n generators[1]. Applying the free differential calculus to elements of Aut (F_n) sometimes gives rise to linear representations of B_n or some of its subgroups. The Burau and Krammer's representations arise this way. It has been shown that the Burau representation of B_n is not faithful for n≥6[4]. For n = 3, it was proved that the Burau representation is indeed faithful[1].

The representation, introduced by D. Krammer, is the map $K(q,t) : B_n \to GL(m, Z[q^{\pm 1}, t^{\pm 1}])$, where m= n (n-1)/2 and q,t are two variables. What distinquishes this representation from others is that Krammer's representation is a faithful representation for all n≥3[3]. In our study, we consider the braid group on three strings and we specialize the indeterminates q and t to non zero complex numbers. Our main theorem, Theorem 5, gives a necessary and sufficient condition for the specialization of Krammer's representation of B_3 to be irreducible.

MATERIALS AND METHODS

Definition 1[1]: The braid group on n strings, B_n, is the abstract group with presentation B_n =

$$\langle \sigma_1,\ldots,\sigma_{n-1} / \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1,\ldots, n-2, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \rangle$$

The generators $\sigma_1,\ldots,\sigma_{n-1}$ are called the standard generators of B_n.

Let us recall the Lawrence-Krammer representation of braid groups[3]. This is a representation of B_n in $GL_m(Z[t^{\pm 1}, q^{\pm 1}]) = Aut(V_0)$, where m = n (n-1)/2 and V_0 is the free module of rank m over $Z[t^{\pm 1}, q^{\pm 1}]$. The representation is denoted by K (q,t). For simplicity, we write K instead of K(q, t).

Definition 2[3]: With respect to $\{x_{i,j}\}_{1\leq i<j\leq n}$, the free basis of V_0, the image of each Artin generator under Krammer's representation is written as:

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$$K(\sigma_k)(x_{i,j}) = \begin{cases} tq^2 x_{k,k+1}, & i=k, j=k+1; \\ (1-q)x_{i,k} + qx_{i,k+1}, & j=k, i<k; \\ x_{i,k} + tq^{k-i+1}(q-1)x_{k,k+1}, & j=k+1, i<k; \\ tq(q-1)x_{k,k+1} + qx_{k+1,j}, & i=k, k+1<j; \\ x_{k,j} + (1-q)x_{k+1,j}, & i=k+1, k+1<j; \\ x_{i,j}, & i<j<k \quad \text{or} \quad k+1<i<j; \\ x_{i,j} + tq^{k-i}(q-1)^2 x_{k,k+1}, & i<k<k+1<j \end{cases}$$

Using the Magnus representation of subgroups of the automorphism group of a free group with three generators, we determine Krammer's representation $K(q,t)\colon B_3 \to GL(3, \mathbb{Z}[q^{\pm 1}, t^{\pm 1}])$, where:

$$K(\sigma_1) = \begin{pmatrix} tq^2 & 0 & 0 \\ tq(q-1) & 0 & q \\ 0 & 1 & 1-q \end{pmatrix}$$

and

$$K(\sigma_2) = \begin{pmatrix} 1-q & q & 0 \\ 1 & 0 & tq^2(q-1) \\ 0 & 0 & tq^2 \end{pmatrix}$$

Here $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ is the ring of Laurent polynomials on two variables. Specializing t and q to non zero complex numbers, we consider the complex linear representation $K(q,t)\colon B_3 \to GL(3,C)$. We show that the only non-zero invariant subspace under the action of the specialization of Krammer's representation of B_3 coincides with the vector space C^3. Here, we regard $M_3(C)$ as acting from the left on column vectors so that eigenvectors and invariant subspaces lie in C^3.

RESULTS

In this section, we find a necessary and sufficient condition for the irreducibility of Krammer's representation of B_3.

Theorem 3: For $(q,t) \in (C^*)^2$, Krammer's representation $K(q,t)\colon B_3 \to GL(3,C)$ is irreducible if $t \neq -1$, $q^3 t \neq 1$ and $q^3 t^2 \neq 1$.

Proof: For simplicity, we write K(a) to denote K(q, t)(a), where $a \in B_3$. We consider the matrix that corresponds to the image of the element $\sigma_1\sigma_2\sigma_1$ under Krammer's representation. Direct computations show that:

$$K(\sigma_1\sigma_2\sigma_1) = \begin{pmatrix} 0 & 0 & q^4 t \\ 0 & q^3 t & 0 \\ q^2 t & 0 & 0 \end{pmatrix}$$

The eigenvalues of $K(\sigma_1\sigma_2\sigma_1)$ are $-q^3t$, q^3t and q^3t. Let us diagonalize the matrix corresponding to this element by an invertible matrix, say T and conjugate the matrices $K(\sigma_1)$, $K(\sigma_2)$ and $K(\sigma_1^2)$ by the same matrix T. The invertible matrix T is given by:

$$T = \begin{pmatrix} -q & q & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

In fact, a computation shows that:

$$T^{-1}K(\sigma_1\sigma_2\sigma_1)T = \begin{pmatrix} -q^3 t & 0 & 0 \\ 0 & q^3 t & 0 \\ 0 & 0 & q^3 t \end{pmatrix}$$

After conjugation by, we get that:

$$T^{-1}K(\sigma_1)T = \frac{1}{2}\begin{pmatrix} 1-q+q^2t & 1-q-q^2t & 1 \\ 1-q-q^2t & 1-q+q^2t & 1 \\ -2q(-1-qt+q^2t) & 2q(1-qt+q^2t) & 0 \end{pmatrix}$$

$$T^{-1}K(\sigma_2)T = \frac{1}{2}\begin{pmatrix} 1-q+q^2t & -1+q+q^2t & -1 \\ -1+q+q^2t & 1-q+q^2t & 1 \\ 2q(-1-qt+q^2t) & 2q(1-qt+q^2t) & 0 \end{pmatrix}$$

and

$$T^{-1}K(\sigma_1{}^2)T = \frac{1}{2}\begin{bmatrix} 1-q+q^2+q^2t-q^3t+q^4t^2 & 1-q+q^2-q^2t+q^3t-q^4t^2 & 1-q \\ 1-q+q^2+q^2t-q^3t-q^4t^2 & 1-q+q^2-q^2t+q^3t+q^4t^2 & 1-q \\ -2q(q-1)(1+q^3t^2) & 2q(q-1)(q^3t^2-1) & 2q \end{bmatrix}.$$

For simplicity, we still call $T^{-1}K(\sigma_1\sigma_2\sigma_1)T$ by $K(\sigma_1\sigma_2\sigma_1)$, $T^{-1}K(\sigma_1)T$ by $K(\sigma_1)$ $T^{-1}K(\sigma_2)T$ by $K(\sigma_2)$ and $T^{-1}K(\sigma_1{}^2)T$ by $K(\sigma_1{}^2)$.

Now, suppose that S is a non-zero invariant subspace of the matrices $K(\sigma_1)$, $K(\sigma_2)$ and $K(\sigma_1{}^2)$. We show, under the conditions of the hypothesis, that the subspace S becomes the vector space C^3 spanned by the standard unit vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

From the diagonal form of $K(\sigma_1\sigma_2\sigma_1)$, we see that the subspace S contains at least one of e_1 or ue_2+ve_3, where $(u, v) \neq (0, 0)$. We consider the following two cases:

Case 1: Assume that $e_1 \in S$. Then we have that $K(\sigma_1)(e_1) \in S$, which implies that:

$$(1-q-q^2t)e_2 - 2q(-1-qt+q^2t)e_3 \in S \quad (1)$$

Also, we have that $K(\sigma_1^2)(e_1) \in S$, which implies that:

$$(1-q+q^2+q^2t-q^3t-q^4t^2)e_2-2q(q-1)(q^3t^2+1)e_3 \in S \quad (2)$$

Notice that if $1-q-q^2t = 0$ then $-1-qt+q^2t \neq 0$ using the hypothesis and so $e_3 \in S$. Likewise, if $1-q+q^2+q^2t-q^3t-q^4t^2 = 0$ then $(q-1)(q^3t^2+1) \neq 0$ and so $e_3 \in S$.

Thus, we may assume that $1-q-q^2t \neq 0$ and $1-q+q^2+q^2t-q^3t-q^4t^2 \neq 0$. (1) and (2) imply that $-2q^2(1+t)(q^3t-1)e_3 \in S$ and so, by our hypothesis, we get that:

$$e_3 \in S$$

Having proved that $e_3 \in S$, we have that $K(\sigma_1)(e_3) \in S$. This implies that $e_2 \in S$. Hence, we conclude that $S = C^3$.

Case 2: Next we assume that $ue_2+ve_3 \in S$ where $(u, v) \neq (0, 0)$. Again, we have that $K(\sigma_1)(ue_2+ve_3) \in S$, which implies that:

$$[(1-q-q^2t)u+v]e_1 + [(1-q+q^2t)u+v]e_2 + 2q(1-qt+q^2t)ue_3 \in S \quad (3)$$

Likewise, we have that $K(\sigma_2)(ue_2+ve_3) \in S$, which implies that:

$$[(-1+q+q^2t)u-v]e_1+[(1-q+q^2t)u+v]e_2+2q(1-qt+q^2t)ue_3 \in S \quad (4)$$

Subtracting (4) from (3), we get that $[(1-q-q^2t)u+v]e_1 \in S$.

If $(1-q-q^2t)u+v \neq 0$ then $e_1 \in S$ and so we apply case 1.

If $(1-q-q^2t)u+v = 0$ then (3) implies that $(qt)e_2+(1-qt+q^2t)e_3 \in S$.

Having that $e_2+(-1+q+q^2t)e_3 \in S$, we get that $(1-q^3t^2)e_3 \in S$. By our hypothesis, we get that $e_3 \in S$. It follows that e_2 and e_1 are also in S. Therefore, as in case 1, we get that $S = C^3$.

Next, our purpose is to find a necessary and sufficient condition that guarantees the irreducibility of the complex specialization of Krammer's representation of B_3. We will show that the condition in Theorem 3 stands as a necessary condition for irreducibility as well. Therefore, we present our next theorem.

Theorem 4: The complex specialization of Krammer's representation $K(q, t): B_3 \to GL(3, C)$ is reducible under any of the following conditions:

- $t = -1$
- $q^3t = 1$
- $q^3t^2 = 1$

Proof: Under each of the conditions of our hypothesis, we will find an invariant subspace under the action of the complex specialization of Krammer's representation of B_3. Recall that the matrices $K(\sigma_1)$ and $K(\sigma_2)$ that will be used in the proof are the ones given in Definition 2:

- Assume that $t = -1$. Consider the two cases whether or not $q^2 = -1$

Case 1: If $q^2 \neq -1$ then we take the invariant subspace as the one generated by the eigenvectors of $K(\sigma_1)$, namely, m and n. Here $m = (0, q, 1)^T$ and $n = (-(q^2+1), -q^2+q-1, 1)^T$, where, T is the transpose. More precisely, we have that $K(\sigma_2)(m) = -\frac{q^4}{q^2+1}m - \frac{q^2}{q^2+1}n$, $K(\sigma_2)(n) = -(q^2+\frac{1}{q^2+1})m + \frac{1}{q^2+1}n$.

Case 2: If $q^2 = -1$ then we take the invariant subspace to be the one generated by $m = (0, q, 1)^T$ and $B = (-1, -1, 0)^T$. To see this:

$$K(\sigma_1)(m) = m,\ K(\sigma_1)(B) = B-m,\ K(\sigma_2)(m) = B+m,$$
$$K(\sigma_2)(B) = B$$

- Assume that $q^3t = 1$. If $q^2t = 1$ then $q = 1 = t$ and so the subspace generated by $(1, 1, 1)^T$ is invariant. Without loss of generality, we assume that $q^2t \neq 1$. Here, we consider the two cases whether or not $qt = -1$.

Case 1: If $qt \neq -1$ then we take the invariant subspace to be the one generated by the eigenvectors of $K(\sigma_1)$, namely $m = (0, -1, 1)^T$, $n = \left(\frac{(q^2t-1)(qt+1)}{t(q-1)}, tq^2+q-1, 1\right)^T$. To see this:

$$K(\sigma_2)(m) = \left(q^2t+\frac{qt(q-1)}{(q^2t-1)(tq+1)}\right)m-\frac{qt(q-1)}{(q^2t-1)(tq+1)}n$$

and

$$K(\sigma_2)(n) = \left(q^2t+\frac{1-q}{(q^2t-1)(tq+1)}\right)m+\frac{q-1}{(q^2t-1)(tq+1)}n$$

Case 2: If $qt = -1$ then we take the invariant subspace to be the one generated by $m = (0, -1, 1)^T$ and $n = (1, q, 0)^T$. To see this:

$$K(\sigma_1)(m) = -qm,\ K(\sigma_1)(n) = \frac{1}{q}(n-m),\ K(\sigma_2)(m) = \frac{1}{q}(n+m),\ K(\sigma_2)(n) = -qn$$

- Assume that $q^3t^2 = 1$. We take the 1-dimensional invariant subspace to be the one generated by the vector $n = (q, q^2t+q-1, 1)^T$. This is true because $K(\sigma_1)(n) = (q^2t)n$ and $K(\sigma_2)(n) = (q^2t)n$.

Combining Theorem 3 and Theorem 4, we obtain our main theorem.

Theorem 5: For $(q, t)\in(C^*)^2$, the specialization of Krammer's representation $K(q,t)$: $B_3 \rightarrow GL(3, C)$ is irreducible if and only if $t\neq -1, q^3t \neq 1$ and $q^3t^2 \neq 1$.

DISCUSSION

So far in the literature, a criterion for the irreducibility of linear representations of the braid group, B_n, of degree n-1 was found. Our goal was to extend this work to Krammer's representation of higher degree, namely, $n(n-1)/2$. Our main result is a partial result that gives a criterion for the irreducibility of Krammer's representation only in the case $n = 3$.

CONCLUSION

We have determined the irreducible complex specializations of the faithful Krammer's representations of the braid group, B_3. A future work is to try to characterize all irreducible Krammer's representations of B_n for any value of n.

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