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Estimation of the Parameters of the Stochastic Differential Equations Black-Scholes Model Share Price of Gold

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Abstract: Problem statement: The estimation of the parameters is one of main problems of the dynamic models in many scientific fields and particularly in economics and finance. In this study, we examine the techniques of estimation of the parameters of the Black-Scholes model. These techniques are based on the function of probability. **Approach:** The two estimations are based on the likelihood function. The "discret" method considers the function of density of transition from the process of diffusion normal log. The second method proposes the estimate of the parameters of the model via the observation of the time of first passage of the process through a constant limit of which the density is known. **Results:** One treats an application of the share price of gold. **Conclusion:** A comparative study between both methods of estimations of the parameters and the forecast is given.

Key words: Density of transition, time of first passage, share price of gold

INTRODUCTION

The most recognized study in the mathematical financial world is certainly the Black and Scholes (Black and Scholes, 1973) option pricing model, which had an immediate success with researchers and professionals in finance, both need was great risk management tool. The Black-Scholes model used to evaluate the pricing of an option in the case of non-arbitration, while being based on the assumption that the evolution of a title follows a geometrical Brownian motion and that volatility is constant (Lamberton and Lapeyre, 2007; Steele, 2000). The Black-Scholes model is, undoubtedly, the first theoretical model evaluation to have been used so extensively by professionals for evaluation, speculation or hedging. However, the first problem, to which one is confronted and who concerns the models in general is that parameter estimation (Agunbiade and Iyaniwura, 2010).

We present the Black-Scholes model, dealing in detail the various techniques for estimating parameters based on the likelihood function (Janssen *et al*., 1996; Gross, 2006; Al Omari *et al*., 2010; Anton *et al*., 2009). The "discret" method considers the density function of the transition process log normal distribution. The second proposes the estimation of the parameters of the model via the observation of the time of first passage of the process through a constant limit of which the density is known. A comparison is made on an application of the share price of gold.

MATERIALS AND METHODS

Estimate of parameters: The model suggested by Black-Scholes describes the behavior of price is a model of continuous time with a risky assets and a not risky assets. We suppose that the behavior of the stock price is determined by the stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{1}$$

where, $B = (B(t), t \geq 0)$ is a standard Brownian motion and μ and σ are real parameters. The change of variables $Y_t = \log(X_t)$ and $m = \left(\mu - \frac{1}{2}\sigma^2\right)$ and the application of the Ito's lemma give:

$$X_t = x_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} \tag{2}$$

This process is known as geometric Brownian or process "log normal". It follows that $Y = (Y(t), t \geq 0)$ is a generalized Wiener's process of drift m and variance:

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$\sigma^2, \forall s, 0 \le s \le t, B_x(t) - B_x(s) \sim N\left(m(t-s), \sigma^2(t-s)\right)$

Therefore:

$$Y(t) - Y(0) = \log\left(X(t)/X(0)\right) \sim N\left(mt, \sigma^2 t\right)$$

The historical method is a statistical method which uses the property of independence and normality logarithmic returns. One has $R_j = \log\left(X_t / X_{t-1}\right),\ j \ge 1$. A very important constraint of the Black-Scholes model is that the returns $R_j, 1 \le j \le n$ are independent and have the same Gaussian law. One notes $m = \left(\mu - \frac{1}{2}\sigma^2\right)$. If there are *n* observations $(r_1, ..., r_n)$ of the returns R_j, the density of this sample, attachment likelihood is given by the product of the Gaussians densities. The estimates of μ and σ are given by:

$$\hat{m} = \hat{\mu} - \frac{1}{2}\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} r_i = \hat{r}$$

or:

$$\hat{\mu} = \bar{r} + \frac{1}{2}\hat{\sigma}^2 \quad (3)$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(r - \hat{m})^2 = s^2 \quad (4)$$

The laws of the random variables of the estimators above are respectively $N\left(m, \frac{\sigma^2}{n}\right)$ and $\frac{\sigma^2}{n}\chi^2_{n-1}$.

Discret method: X(t) is a process of diffusion log normal characterized by the function of density of following transition, for s≤t, X(s) = y:

$$f(x, t / y, s) = \frac{1}{\sigma x \sqrt{2\Pi}} \exp\left\{-\frac{1}{2\sigma^2(t-s)}\left[\log x - \log y - \left(\mu - \frac{1}{2}\sigma^2\right)(t-s)\right]^2\right\} \quad (5)$$

We consider b independent paths of the process $X, (i = 1, ..., b)$ and by applying the method of maximum likehood from the density (5), the resolution of the equations of probability gives us the following estimators:

$$\hat{\mu} = \frac{\sum_{i=1}^{b}\log\left(\frac{x_{in}}{x_{i0}}\right)}{\sum_{i=1}^{b}\left(t_{in_i} - t_{i10}\right)} + \frac{1}{2}\hat{\sigma}^2$$

and:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{b}\sum_{j=1}^{n_i}\left(t_{ij} - t_{ij-1}\right)\left[\frac{\log\left(\frac{x_{ij}}{x_{ij-1}}\right)}{t_{ij} - t_{ij-1}} - \frac{\sum_{i=1}^{b}\log\left(\frac{x_{in_i}}{x_{i0}}\right)}{\sum_{i=1}^{b}\left(t_{in_i t_{i0}}\right)}\right]^2}{k}$$

where, $k = \sum_{i=1}^{b} n_i$ represents the total number of observations. We have the asymptotic laws $\sqrt{\sum\left(t_{in_i - t_{i0}}\right)}\frac{(\hat{m} - m)}{\sigma} \sim N(0,1) \sim$ and $k\frac{\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{k-1}$.

Method via Time of First Passage (TFP): T is the first time of passage of the process $X(t) = \exp\left(Y(t)\right)$ by the point $S = \exp(a)$. We have: $T = \inf\{t; X(t) = S\}, t > 0$. The first time T follows an inverse Gaussian distribution, its density function, given $X(t_0) = x_0 \sim$, therefore written as:

$$f(x, t / y, s) = \frac{\log\left(\frac{S}{x_0}\right)}{\sqrt{2\Pi\sigma\left(t - t_0\right)^{\frac{3}{2}}}} \exp\left\{-\frac{1}{2\sigma^2\left(t - t_0\right)}\left[\log\left(\frac{S}{x_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\left(t - t_0\right)\right]^2\right\} \quad (6)$$

The random variables $T_1, ..., T_b$ are the moments of the first time of passage through the constant limit *S* in the b trajectories, with observed values $t_i : i = 1, ..., b$. The resolution of the equations of probability for the density (6) gives the following estimators:

$$\hat{\mu} = \frac{\log\left(\frac{S}{x_0}\right)}{\ddot{t}} + \frac{1}{2}\hat{\sigma}^2$$

and:

$$\hat{\sigma}^2 = \frac{\left[\log\left(\frac{S}{x_0}\right)\right]^2}{b}\sum_{i=1}^{b}\left(\frac{1}{t_i} - \frac{1}{t}\right)$$

The asymptotic laws are respectively $H=\frac{\sqrt{b}\left(m\ddot{T}-a\right)}{b}\sim N(0,1)$ and $k\frac{\hat{\sigma}^2}{\sigma^2}\sim\chi^2_{k-1}$.

RESULTS

Application in finance: It has 140 daily observations of the share price of gold for the period from April 2-December 31, 2007, whose evolution in USD is represented on the Fig. 1.

The Box Ljung test so that the Jarque Bera test allowing to assert that returns, Fig. 2, are independent and that the assumption of normality is accepted.

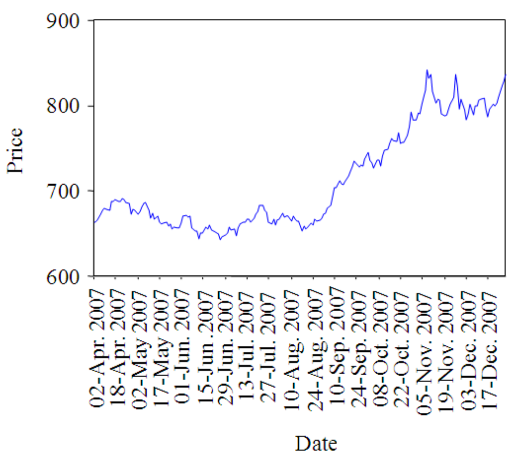

Fig. 1: Evolution of share of gold

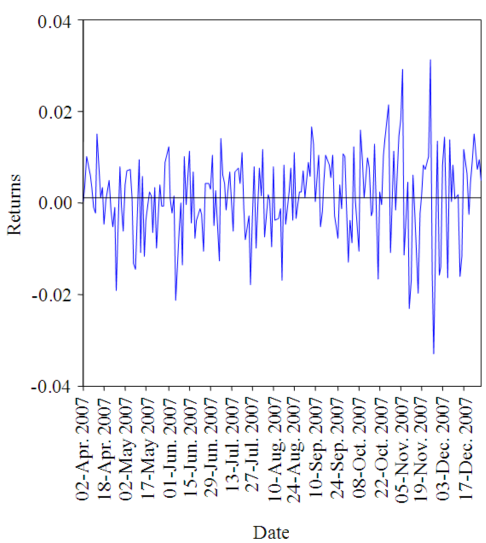

Fig. 2: Evolution of returns

The estimates of μ and σ and the historical method, depend only on the observations of the returns $\left(r_1,\ldots,r_n\right)$ and are given by:

$$\hat{\mu}=\ddot{r}+\frac{1}{2}\hat{\sigma}^2=0.0013 \text{ and } \ddot{\sigma}^2=\sum_{i=1}^{n}\left(r_i-\ddot{r}\right)^2=s^2=0.0009$$

Estimate by the discret method and method TFP: In finance, the trajectory of the stock price is unique and to apply the two methods of estimate, one considers several paths from only one. For the discret method, one subdivides the observation period in several intervals of the same time length h; thus the starting point is not necessarily the same for all paths.

The construction of the paths in the second method of estimation is done in a following way:

- To choose an initial value which will share the path in several small paths having all the same starting point. This implies that $X_{i0}=x_0$, $\left(i=1,\ldots,b\right)$
- To choose a constant limit *S* such as this terminal is well reached by all the possible trajectories

One distinguishes two cases for the position from *S* relative to x_0. If m>0 one takes in this case, a barrier *S* above $x_0\left(S>x_0\right)$ and when $m<0$, $\left(S<x_0\right)$.

In our series, one can fix at x_0 = 642.85 sharing the series of share price of gold in two trajectories (b = 2) and a constant limit S, such as these trajectories are reached the terminal S. One can find several limits constant S for an x_0 fixed. The best terminal is that which gives a small error with the true values of the share price of gold.

In our application, the limit S = 663 gives the best results (Table 1).

Table 1: Best terminal

S/x_0	$\hat{\mu}$	$\hat{\sigma}^2$	Average error (%)	Quadratic error
661	$5.5238\ 10^{-4}$	$2.0950\ 10^{-6}$	4.47	5.92
663	$6.128\ 10^{-4}$	$3.3053\ 10^{-6}$	3.02	3.87
664	$9.757\ 10^{-4}$	$1.5313\ 10^{-4}$	32.72	42.65

Table 2: The criterion of the minimal variance

	Discret method	Method via TFP
$\hat{\mu}$	0.0012	0.0024
$\hat{\sigma}^2$	8.925×10^{-5}	2.3592×10^{-5}
IC $(\hat{\mu})$	[0.000, 0.0018]	[0.0000, 0.0157]
IC ($\hat{\sigma}^2$)	[0.982, 0.8990]	
	[0.0000, 0.0095] $\times10^{-3}$	
Average error (%)	21.89	3.02
Quadratic error	27.77	3.87

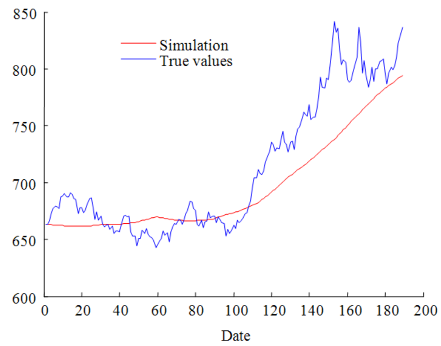

Fig. 3: Simulation and true values

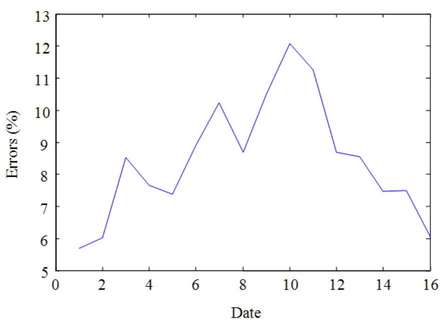

Fig. 4: Relative errors of the forecast

According to the criterion of the minimal variance (Table 2) and the study of the errors, the TFP method offers the better results (3.02%) than those of the discret method (21.89%).

Simulation and forecast: We take the parameters $\hat{\mu}$ and $\hat{\sigma}^2$ estimated by second method TFP to the simulation and forecast period is 15 days.

The graph of simulation and true values for the share of gold is represented in Fig. 3.

The graph of the relative errors of the forecast (Fig. 4) reveals us that this error does not exceed 12% and the relative average error equalizes has 8.44%.

DISCUSSION

The developpement of the methods of the estimation of the parameters can give better results on the main result which is the satisfactory forecast obtained in the short term.

CONCLUSION

We have devoted our study to apply statistical methods to stochastic differential equations, initially to estimate by the historical method, which uses the property of independence and normality of the outputs. The Black-Scholes model and its alternatives are largely used by the professionals. For that, the estimate of its parameters deserves that we interested in other techniques more adapted: Discret method and method via time of the first passage.

The discret method makes it possible to estimate the parameters of Black-Scholes model in the case of the discrete paths. In this method, it is necessary to observe the process during a certain interval of time i.e., to use all the observations of the paths. In the second technique, method via time of the first passage, it is not necessary to observe the process during intervals of times given or to know the evolution of the process until a certain stopping time.

The estimates of the method of the first time of passage gives better results that the discret method while being based on the criterion which minimizes the variances of the estimators and the small errors with the true values of the share price of gold

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