Journal of Mathematics and Statistics 7 (1): 28-36, 2011
ISSN 1549-3644

On Computing of Eigenvalues of Differential Equations $Q = \lambda P$ with Eigenparameter in Boundary Conditions

S.M. Al-Harbi
Department of Mathematics, University College, Umm Al-Qura University,
Makkah, Saudi Arabia, P.O. Box 8140

Abstract: Problem statement: Our purpose of this study is to use sinc methods to compute approximately the eigenval-ues of second-order operator pencil of the form Q-λP. **Approach:** Where Q is second order self adjoint differential operator and P is a first order and λ∈C is an eigenvalue parameter. **Results:** The eigenparameter appears in the boundary conditions linearly. Using computable error boundswe obtain eigenvalue enclosures in a simple way. **Conclusion/Recommendations:** We give some numerical examples and makecompanions with existing results.

Key words: Sinc method, operator pencil, eigenvalue problem, eigenparameter in boundary conditions, computing eigenvalues, Whittaker-Kotel'nikov-Shannon (WKS)

INTRODUCTION

The aim of the present study is to compute the eigenvalues numerically of a differential operator of the form Q-λP approximately by the sinc method, where Q and P are self-adjoint differential operators of the second and first order respectively. By the sinc method we mean the use of the Whittaker-Kotel'nikov-Shannon (WKS) sampling theorem, (Shannon, 1949; Whittaker, 1915; Zayed, 1993). The WKS states that if f(λ) is entire in λ of exponential type σ, σ>0, which belongs to L2(R) where restricted to $\mathbb{R}$, then f(λ) can be reconstructed via the sampling representation:

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\sigma}\right)\operatorname{sinc}(\sigma\lambda - n\pi), \lambda \in \mathbb{C} \qquad (1.1)$$

Series (1.1) converges absolutely on $\mathbb{C}$ and uniformly on $\mathbb{R}$ and on compact subsets of $\mathbb{C}$ (Butzer *et al*., 2001; Stenger, 1993). The space of all such f is the Paley-Wiener space of band limited functions with band width σ which will be denoted by PW_σ^2. The nodes $\left\{\frac{n\pi}{\sigma}\right\}_{n\in\mathbb{Z}}$ are called the sampling points and the sinc functions are:

$$\operatorname{sinc}(\sigma\lambda - n\pi) := \begin{cases} \dfrac{\operatorname{sinc}(\sigma\lambda - n\pi)}{(\sigma\lambda - n\pi)}, & \lambda \neq \dfrac{n\pi}{\sigma} \\ 1, & \lambda = \dfrac{n\pi}{\sigma} \end{cases} \qquad (1.2)$$

$n \in \mathbb{Z}$ Theorem (1.1) is used extensively in approximating solutions and eigenvalues of boundary value problems, (Boumenir, 2000a; 2000b; Lund and Bowers, 1992; Stenger, 1981; 1993). One type of error is associated with sinc-based methods, truncation error. An estimate for the truncation error is established by Jagerman (1966), as follows. For $N \in \mathbb{N}$ and $f(\lambda \in PW_\sigma^2)$, let $f_N(\lambda)$ be the truncated cardinal series:

$$f_N(\lambda) := \sum_{n=-N}^{N} f\left(\frac{n\pi}{\sigma}\right)\operatorname{sinc}(\sigma\lambda - n\pi) \qquad (1.3)$$

Jagerman proved that if $\lambda \in \mathbb{R}$ and in addition $\lambda^k f(\lambda) \in L^2(\mathbb{R})$, for some integer k>0, then for $N \in \mathbb{N}, |\lambda| < N\pi/\sigma$, we have:

$$|f(\lambda) - f_N(\lambda)| \le \frac{E_k(f)|\sin\sigma\lambda|}{\pi(\pi/\sigma)^k\sqrt{1-4^{-k}}}\left(\frac{1}{\sqrt{N\pi/\sigma - \lambda}} + \frac{1}{\sqrt{N\pi/\sigma + \lambda}}\right) \qquad (1.4)$$

$$\frac{1}{(N+1)^k}, \lambda \in \mathbb{R}$$

Where:

$$E_k(f) := \left\{\int_{-\infty}^{\infty} \lambda^{2k} |f(\lambda)|^2 dt\right\}^{1/2} \qquad (1.5)$$

We are concerned with the computation of eigenvalues of the boundary-value problem:

$$-(p(y'-ry))'(x)-\bar{r}(x)p(x)(y'-ry)(x)+ q(x)y(x)=\lambda(2i\rho y'+i\rho' y+wy)(x),\ 0\le x\le 1 \quad (1.6)$$

$$\cos\gamma y(0)-\sin\gamma(p(y'-ry)+i\lambda\rho y)(0)=0 \quad (1.7)$$

$$\cos\delta y(1)+\sin\delta(p(y'-ry)+i\lambda\rho y)(1)=0 \quad (1.8)$$

where, p, q, ρ and w are real-valued functions on [0, 1], p^{-1}, r, q, $w\in L^1(0, 1)$, p≥0, ρ∈AC[0, 1], the set of all absolutely continuous functions on [0, 1], q is essentially bounded from below and γ, δ∈[0,π). This problem has been studied in its general form in the comprehensive study of (Langer *et al*., 1966) as a linear pencil Q-λP, where Q is a second-order operator and P is a first-order operator. Problem (1.6-1.8) differs from classical second-order eigenvalue problems in several respects. First, the operator in the left-hand side of (1.6) is not the identity operator multiplied by the eigenparameter, but a first order operator. Also, the eigenvalue parameter appears linearly in the boundary conditions. Illustrative examples and tables are included in the last section. It is worthy to mention that the sampling scheme is used to approximate eigenvalues for different types of boundary value problems in (Boumenir, 1999; 2000a; 2000b; Chanane, 1999; 2005).

Preliminaries: In the following we consider the eigenvalue problem (1.6-1.8) introduced in Section 1 above. For simplicity, we assume that $\gamma,\delta\in[0,\frac{\pi}{2}]$ and without any loss of generality, we assume that r = 0, p = ρ = w = 1 on [0, 1]. Thus we consider the eigenvalue problem:

$$-y''(x,\lambda)+q(x)y(x,\lambda)=\lambda(2iy'(x,\lambda))0\le x\le 1 \quad (2.1)$$

$$U_1(y):=\cos\gamma y(0,\lambda)-\sin\gamma(y'(0,\lambda)+i\lambda y(0,\lambda))=0 \quad (2.2)$$

$$U_2(y):=\cos\delta y(1,\lambda)+\sin\delta(y'(1,\lambda)+i\lambda y(0,\lambda))=0 \quad (2.3)$$

where, $q\in L^1(0, 1)$. Let φ(., λ) denote the solution of (2.1) satisfying the following initial conditions:

$$\varphi(0,\lambda)=\sin\gamma,\ \varphi^{[1]}(0,\lambda)=\cos\gamma \quad (2.4)$$

where, $\varphi^{[1]}(x,\lambda) := \varphi'(x,\lambda)+i\lambda\varphi(x,\lambda)$. Since φ(.,λ) satisfies (2.2), then the eigenvalues of problem (2.1-2.3) are the zeros of the function, cf. (Langer *et al*., 1966):

$$\Omega(\lambda):=e^{2i\lambda}[\cos\delta\varphi(1,\lambda)+\sin\delta\varphi^{[1]}(1,\lambda)]=e^{2i\lambda}H(\lambda) \quad (2.5)$$

Where:

$$\Delta(\lambda):=[\cos\delta\varphi(1,\lambda)+\sin\delta\varphi^{[1]}(1,\lambda)] \quad (2.6)$$

According to (Langer *et al*., 1966) Δ(λ) has two sequences of positive and negative simple eigenvalues $\{\lambda_k^{\pm}\}_{k=1}^{\infty}$. Using the method of variation of constants, the solution φ(x, λ) satisfies the integral equation:

$$\varphi(x,\lambda)=e^{-i\lambda x}\left[\sin\gamma\cos\sqrt{\lambda^2+\lambda}x+\cos\gamma\frac{\sin\sqrt{\lambda^2+\lambda}x}{\sqrt{\lambda^2+\lambda}}\right]+T_\lambda\varphi(x,\gamma) \quad (2.7)$$

where, T_λ is the Volterra integral operator:

$$T_\lambda f(x)=\int_0^x\frac{\sin\sqrt{\lambda^2+\lambda}(x-t)}{\sqrt{\lambda^2+\lambda}}e^{-i\gamma(x-t)}q(t)f(t)dt \quad (2.8)$$

Differentiating (2.7) and adding the result to iλφ(x, λ), we obtain:

$$\varphi^{[1]}(x,\lambda)=e^{-i\lambda x}[-\sqrt{\lambda^2+\lambda}\sin\gamma\sin\sqrt{\lambda^2+\lambda}x+\cos\gamma\cos\sqrt{\lambda^2+\lambda}x]+\tilde{T}_\lambda\varphi(x,\lambda) \quad (2.9)$$

Here $\tilde{T}_\lambda$ is the Volterra integral operator:

$$\tilde{T}_\lambda f(x)=\int_0^x\cos\sqrt{\lambda^2+\lambda}(x-t)e^{-i\lambda(x-t)}q(t)f(t)dt \quad (2.10)$$

Define u(x, λ) and v(x, λ) to be:

$$u(x,\lambda):=T_\lambda\varphi(x,\lambda),\ v(x,\lambda):=\tilde{T}_\lambda\varphi(x,\lambda) \quad (2.11)$$

In the following, we shall make use of the estimates (Chadan and Sabatier, 1989):

$$\left|\sqrt{\lambda+\mu}\right|\le\sqrt{|\lambda|}+\sqrt{|\mu|},\ |\cos z|\le e^{|\Im z|},\left|\frac{\sin z}{z}\right|\le\frac{c_0}{1+|z|}e^{|\Im z|} \quad (2.12)$$

where, c_0 is some constant (we may take $c_0 \simeq 1.72$ cf. (Chadan and Sabatier, 1989)). For convenience, we define the constants:

$$\begin{aligned}&q_0=\int_0^1|q(t)|dt,\ c_4=|\sin\gamma|+c_0|\cos\gamma|,\ c_5=c_0q_0\\&c_1=c_4\exp c_5,\ \ c_2=c_1q_0,\ c_3=c_1c_5\\&|\cos\delta|+(q_0c_4+c_2c_5)|\sin\delta|\end{aligned} \quad (2.13)$$

Table 1: Observe that $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ when N = 40, m = 10 and θ = 1/15

lk	Exact λ_k	a_-	a_+	$\lambda_{k,N}$
λ_{-2}	-6.842613403785793	-6.8542419744969810	-6.8299202670083460	-6.8426134037858070
λ_{-1}	-3.758605094490024	-3.7632697348613644	-3.7541208431839883	-3.7586050944900227
λ_0	2.758605094490024	2.7518990040100073	2.7654801295144110	2.7586050944900240
λ_1	5.842613403785775	5.8400058020619420	5.8449689020210150	5.8426134037858010
λ_2	8.964493358995961	8.9241740258191800	9.0026492126459700	8.9644933589959540

From (2.7) and (2.10), we have:

$$u(x,\lambda)=\int_0^x \frac{\sin\sqrt{\lambda^2+\lambda}(x-t)}{\sqrt{\lambda^2+\lambda}}e^{-i\lambda x}q(t)\left[\sin\gamma\cos\sqrt{\lambda^2+\lambda}t+\cos\gamma\frac{\sin\sqrt{\lambda^2+\lambda}t}{\sqrt{\lambda^2+\lambda}}\right]dt \tag{2.14}$$
$$+\int_0^x \frac{\sin\sqrt{\lambda^2+\lambda}(x-t)}{\sqrt{\lambda^2+\lambda}}e^{-i\lambda(x-t)}q(t)u(t,\lambda)dt$$

Lemma 1: For $\lambda\in\mathbb{C}$, the following estimates hold:

$$|u(x,\lambda)|\le\frac{c_1c_5}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x) \tag{2.15}$$

$$|u(x,\lambda)|\le\frac{ec_1c_5}{1+\left|\sqrt{\lambda^2+\lambda}\right|}e^{2|\lambda|x} \tag{2.16}$$

Proof: Using the inequalities (2.12), we have for $\lambda\in\mathbb{C}$:

$$\left|\int_0^x \frac{\sin\sqrt{\lambda^2+\lambda}(x-t)}{\sqrt{\lambda^2+\lambda}}e^{-i\lambda x}q(t)\left[\sin\gamma\cos\sqrt{\lambda^2+\lambda}t+\cos\gamma\frac{\sin\sqrt{\lambda^2+\lambda}t}{\sqrt{\lambda^2+\lambda}}\right]dt\right|$$
$$\le\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\int_0^x|q(t)|\frac{c_0(x-t)}{1+\left|\sqrt{\lambda^2+\lambda}\right|(x-t)}\left[|\sin\gamma|+\frac{c_0|\cos\gamma|t}{1+\left|\sqrt{\lambda^2+\lambda}\right|t}\right]dt$$
$$\le\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\frac{c_0x}{1+\left|\sqrt{\lambda^2+\lambda}\right|x}\int_0^x|q(t)|[|\sin\gamma|+|\cos\gamma|c_0t]dt \tag{2.17}$$
$$\le\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\frac{c_0}{1+\left|\sqrt{\lambda^2+\lambda}\right|x}\int_0^1|q(t)|[|\sin\gamma|+|\cos\gamma|c_0t]dt$$

On the other hand:

$$\left|\int_0^x \frac{\sin\sqrt{\lambda^2+\lambda}(x-t)}{\sqrt{\lambda^2+\lambda}}e^{-i\lambda(x-t)}q(t)u(t,\lambda)dt\right|$$
$$\le\int_0^x\frac{c_0(x-t)}{1+\left|\sqrt{\lambda^2+\lambda}\right|(x-t)}\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)(x-t))|q(t)||u(t,\lambda)|dt \tag{2.18}$$
$$\le c_0\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\int_0^x\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)t)|q(t)||u(t,\lambda)|dt,\lambda\in\mathbb{C}$$

Combining (2.17) and (2.18) together with (2.14), we obtain for any complex λ:

$$|u(x,\lambda)|\le\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\frac{c_0}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\int_0^1|q(t)|[|\sin\gamma|+|\cos\gamma|c_0t]dt \tag{2.19}$$
$$+c_0\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\int_0^x\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)t)|q(t)||u(t,\lambda)|dt$$

The use of Gronwall's inequality, cf. (Eastham, 1970), yields, $\lambda\in\mathbb{C}$:

$$\exp(-(|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)|u(x,\lambda)|\le$$
$$\left[\frac{c_0}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\int_0^1|q(t)|[|\sin\gamma|+|\cos\gamma|c_0t]dt\right]\exp(c_0\int_0^x|q(t)|dt)$$
$$\le\left[\frac{c_0}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\int_0^1|q(t)|[|\sin\gamma|+|\cos\gamma|c_0t]dt\right]\exp(c_0\int_0^1|q(t)|dt)$$

Therefore:

$$|u(x,\lambda)|\le\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\left[\frac{c_0[|\sin\gamma|+|\cos\gamma|c_0]}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\int_0^1|q(t)|dt\right]$$
$$\exp(c_0\int_0^1|q(t)|dt)=\frac{c_1c_5}{1+\left|\sqrt{\lambda^2+\lambda}\right|}\exp((|\Im\lambda|+\left|\Im\sqrt{\lambda^2+\lambda}\right|)x)\lambda\in\mathbb{C}$$

From (2.12), we get, for $\lambda \in \mathbb{C}$:

$$\left|\sqrt{\lambda^2+\lambda}\right| = \left|\sqrt{\left(\lambda+\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)}\right| \le \sqrt{\left|\lambda+\frac{1}{2}\right|^2} + \sqrt{\left|\frac{1}{4}\right|} = \left|\lambda+\frac{1}{2}\right| + \frac{1}{2} \le |\lambda| + 1$$

Then from the previous inequality together with (2.15), we get (2.16).

Also, from (2.7) and (2.11), we have:

$$v(x,\lambda) = \int_0^x \cos\sqrt{\lambda^2+\lambda}(x-t)e^{-i\lambda x}q(t) \left[\sin\gamma\cos\sqrt{\lambda^2+\lambda t} + \cos\gamma\frac{\sin\sqrt{\lambda^2+\lambda t}}{\sqrt{\lambda^2+\lambda}}\right]dt + \int_0^x \cos\sqrt{\lambda^2+\lambda}(x-t)e^{-i\lambda(x-t)}q(t)u(t,\lambda)dt \quad (2.20)$$

Hence, by using (2.12), we have the following estimates.

Lemma 2: For $\lambda \in \mathbb{C}$, the following estimates hold:

$$|v(x,\lambda)| \le (q_0c_4 + c_2c_5)\exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \quad (2.21)$$

$$|v(x,\lambda)| \le e(q_0c_4 + c_2c_5)e^{2|\lambda|x} \quad (2.22)$$

Proof: Using the inequalities (2.12), we have for $\lambda \in \mathbb{C}$:

$$\left|\int_0^x \cos\sqrt{\lambda^2+\lambda}(x-t)e^{-i\lambda x}q(t)\left[\sin\lambda\cos\sqrt{\lambda^2+\lambda} + \cos\gamma\frac{\sin\sqrt{\lambda^2+\lambda t}}{\sqrt{\lambda^2+\lambda}}\right]dt\right|$$
$$\le \exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x)\int_0^x |q(t)| \left[|\sin\gamma| + \frac{c_0|\cos\gamma|t}{1+|\sqrt{\lambda^2+\lambda}|t}\right]dt$$
$$\le \exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \int_0^x |q(t)|[|\sin\gamma| + |\cos\gamma|c_0t]dt \quad (2.23)$$
$$\le \exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \int_0^1 |q(t)|[|\sin\gamma| + |\cos\gamma|c_0t]dt$$

Also, from (2.15), we have:

$$\left|\int_0^x \cos\sqrt{\lambda^2+\lambda}(x-t)e^{-i\lambda}q(t)u(t,\lambda)dt\right| \le \frac{c_2c_5}{1+|\sqrt{\lambda^2+\lambda}|}\exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \quad (2.24)$$

Combining (2.23) and (2.24) together with (2.20), we obtain for any complex λ:

$$|v(x,\lambda)| \le \exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \int_0^1 |q(t)|[|\sin\gamma| + |\cos\gamma|c_0t]dt + \frac{c_2c_5}{1+|\sqrt{\lambda^2+\lambda}|}\exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)x) \quad (2.25)$$

The rest of the proof can be accomplished as in the previous lemma.

The method and error bounds: In this section we derive the method of computing eigenvalues of problem (2.1-2.3) numerically. The basic idea of the scheme is to split $\Delta(\lambda)$ into two parts:

$$\Delta(\lambda) := G(\lambda) + S(\lambda) \quad (3.1)$$

where, $S(\lambda)$ is the unknown part:

$$S(\lambda) := \cos\delta u(1,\lambda) + \sin\delta v(1,\lambda) \quad (3.2)$$

and $G(\Delta)$ is the known part:

$$G(\lambda) := e^{-i\lambda}\left[(\cos\delta\sin\gamma + \sin\delta\cos\gamma)\cos\sqrt{\lambda^2+\lambda} + (\cos\delta\cos\gamma - \sin\delta\sin\gamma(\lambda^2+\lambda))\frac{\sin\sqrt{\lambda^2+\lambda}}{\sqrt{\lambda^2+\lambda}}\right] \quad (3.3)$$

Then, from Lemma 2.1 and Lemma 2.2, we have the following lemma.

Lemma 3: The function $S(\lambda)$ is entire in λ for each $x \in [0, 1]$ and the following estimates hold:

$$|S(\lambda)| \le c_3\exp((|\Im\lambda| + |\Im\sqrt{\lambda^2+\lambda}|)) \quad (3.4)$$

$$|S(\lambda)| \le ec_3e^{2|\lambda|} \quad (3.5)$$

Proof: Since:

$$|S(\lambda)| \le |\cos\delta||u(1,\lambda)| + |\sin\delta||v(1,\lambda)| \quad (3.6)$$

Table 2: Observe that $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ when N = 40, m = 20 and θ = 1/10

lk	Exact λ_k	a_-	a_+	$\lambda_{k,N}$
λ_{-2}	-6.842613403785793	-6.8427489879179450	-6.8424777272988150	-6.8426134037858080
λ_{-1}	-3.758605094490024	-3.7586220927015312	-3.7585880937270120	-3.7586050944900236
λ_0	2.758605094490024	2.7585832691014900	2.7586269202341933	2.7586050944900210
λ_1	5.842613403785775	5.8425042541914920	5.8426134037857750	5.8426134037858070
λ_2	8.964493358995961	8.9636764517170560	8.9653127499942840	8.9644933589959380

Table 3: Observe that $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ when N = 30, m = 8 and θ = 1/11

lk	Exact λ_k	a_-	a_+	$\lambda_{k,N}$
λ_{-2}	-3.7419233725545210	-3.74237866791672640	-3.74146912206194000	-3.7419233725545240
λ_{-1}	-1.2582490364604133	-1.25859144578689120	-1.25790866905852680	-1.2582490364604058
λ_0	0.2582490364604128	0.25749078764309463	0.25901052779376754	0.2582490364603885
λ_1	2.7419233725545210	2.74136528771669540	2.74248197201602740	2.7419233725545490
λ_2	5.8305081032590080	5.83042779415377500	5.83058859204998700	5.8305081032590140

then from (2.15) and (2.21), we get:

$$\begin{aligned}|S(\lambda)| \le |\cos\delta| \frac{c_1 c_5}{1+\left|\sqrt{\lambda^2+\lambda}\right|} \exp((|\Im\lambda| + \left|\Im\sqrt{\lambda^2+\lambda}\right|)) + \\ |\sin\delta|(q_0 c_4 + c_2 c_5)\exp((|\Im\lambda| + \left|\Im\sqrt{\lambda^2+\lambda}\right|))\end{aligned} \quad (3.7)$$

leading to (3.4). Also, from (2.16), (2.22) and (3.6) we obtain (3.5).

Let θ∈(0, 1) and $m \in \mathbb{Z}^+$, m≥1 be fixed. Let $G_{\theta,m}(\lambda)$ be the function:

$$F_{\theta,m}(\lambda) := \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^m S(\lambda), \lambda \in \mathbb{C} \quad (3.8)$$

The number θ will be specified latter. The number 1 is the smallest positive integer that suites our investigation as is seen in the next lemma.

Lemma 4: $F_{\theta,m}(\lambda)$ is an entire function of λ which satisfy the estimates:

$$|F_{\theta,m}(\lambda)| \le \frac{c_3 c_0^m}{(1+\theta|\lambda|)^m} \exp((|\Im\lambda|(1+m\theta) + \left|\Im\sqrt{\lambda^2+\lambda}\right|)) \quad (3.9)$$

$$|F_{\theta,m}(\lambda)| \le \frac{e c_3 c_0^m}{(1+\theta|\lambda|)^m} e^{|\lambda|(2+m\theta)} \quad (3.10)$$

Moreover, $\lambda^{m-1}F_{\theta,m}(\lambda) \in L^2(\mathbb{R})$ and:

$$E_{m-1}(F_{\theta,m}) = \sqrt{\int_{-\infty}^{\infty} \left|\lambda^{m-1}F_{\theta,m}(\lambda)\right|^2 d\lambda} \le c_0^m c_3 v_0 \quad (3.11)$$

Where:

$$v_0 := \sqrt{\frac{(1+\theta)^{-2m}(-1+(-1+(1+\frac{1}{\theta})^{2m})\theta)}{2m-1} + 1.489830^{2-2m} + \frac{\theta^{1-2m}}{2m-1}}$$

Proof: Since S(λ) is entire, then also $F_{\theta,m}(\lambda)$ is entire in λ. Combining the estimates $\left|\frac{\sin z}{z}\right| \le \frac{c_0}{1+|z|} e^{|\Im z|}$, where $c_0 \simeq 1.72$, cf. (Chadan and Sabatier, 1989) and (3.4), we obtain:

$$\begin{aligned}|F_{\theta,m}(\lambda)| \le \left(\frac{c_0}{1+\theta|\lambda|}\right)^m e^{|\Im z|m\theta}.c_3 \\ \exp((|\Im\lambda| + \left|\sqrt{\lambda^2+\lambda}\right|)), \lambda \in \mathbb{C}\end{aligned} \quad (3.12)$$

leading to (3.9). Also, as the above lemmas, we can prove (3.10). Therefore if λ∈(-∞, -1)∪(0, ∞), we have:

$$\left|\lambda^{m-1}F_{\theta,m}(\lambda)\right| \le \frac{c_0^m c_3 |\lambda|^{m-1}}{(1+\theta|\lambda|)^m} \quad (3.13)$$

and from which:

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\lambda^{m-1}F_{\theta,m}(\lambda)\right|^2 d\lambda \le c_0^{2m} c_3^2 \left[\int_{-\infty}^{-1} \frac{|\lambda|^{2m-2}}{(1+\theta|\lambda|)^{2m}} d\lambda + \int_{-1}^{0} \frac{|\lambda|^{2m-2} e^{\sqrt{-\lambda^2-\lambda}}}{(1+\theta|\lambda|)^{2m}} d\lambda\right] \\ + \int_0^{\infty} \frac{|\lambda|^{2m-2}}{(1+\theta|\lambda|)^{2m}} d\lambda\end{aligned} \quad (3.14)$$

Then $\lambda^{m-1}F_{\theta,m}(\lambda) \in L^2(\mathbb{R})$ and by calculating the integrals we obtain (3.11).

What we have just proved is that $F_{\theta,m}(\lambda)$ belongs to the Paley-Wiener space PW_σ^2 with σ = 2+mθ.

Table 4: Observe that $\lambda_{k,N}$ and the exact solution λ_k are all inside the interval $[a_-, a_+]$ when N = 30, m = 5 and θ = 2/25

lk	Exact λ_k	a_-	a_+	$\lambda_{k,N}$
λ_{-2}	-3.7419233725545210	-3.7435280774791470	-3.74029081919575300	-3.7419233725312133
λ_{-1}	-1.2582490364604133	-1.2591411072738001	-1.25732395370185630	-1.2582490364748544
λ_0	0.2582490364604128	0.2539975606237622	0.26261020951039676	0.2582490363879327
λ_1	2.7419233725545210	2.7408673846710445	2.74299646377402300	2.7419233725741194
λ_2	5.8305081032590080	5.8283634020522745	5.83265514673283900	5.8305081032150110

Hence, $F_{\theta,m}(\lambda)$ can be recovered from its values at the points $\lambda_n = \frac{n\pi}{\sigma}, n \in \mathbb{Z}$ via the sampling expansion:

$$F_{\theta,m}(\lambda) := \sum_{n=-\infty}^{\infty} F_{\theta,m}\left(\frac{n\pi}{\sigma}\right)\sin(\sigma\lambda - n\pi) \tag{3.15}$$

Let $N \in \mathbb{Z}^+, N > m$ and approximate $F_{\theta,m}(\theta)$ by its truncated series $F_{\theta,m,N}(\lambda)$, where:

$$F_{\theta,m}(\lambda) := \sum_{n=-N}^{N} F_{\theta,m}\left(\frac{n\pi}{\sigma}\right)\sin(\sigma\lambda - n\pi) \tag{3.16}$$

Since $\lambda^{m-1}F_{\theta,m}(\lambda) \in L^2(\mathbb{R})$, the truncation error is given for $|\lambda| < \frac{N\pi}{\sigma}$ by:

$$\left|F_{\theta,m}(\lambda) - F_{\theta,m,N}(\lambda)\right| \le T_N(\lambda) \tag{3.17}$$

Where:

$$T_N(\lambda) := \frac{E_{m-1}(F_{\theta,m})}{\sqrt{1-4^{-m+1}}\pi(\pi/\sigma)^{m-1}} \frac{|\sin\sigma\lambda|}{(N+1)^{m-1}} \left[\frac{1}{\sqrt{N\pi/\sigma - \lambda}} + \frac{1}{\sqrt{N\pi/\sigma + \lambda}}\right] \tag{3.18}$$

Let:

$$\Delta_N(\lambda) + := G(\lambda) + \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m} F_{\theta,m,N}(\lambda)$$

Then (3.17) implies:

$$\left|\Delta(\lambda) - \Delta_N(\lambda)\right| \le \left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m} T_N(\lambda), |\lambda| < \frac{N\pi}{\sigma} \tag{3.19}$$

and θ is chosen sufficiently small for which $|\theta\lambda| < \pi$.

Let λ^* be an eigenvalue, that is:

$$\Delta(\lambda^*) = G(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} F_{\theta,m}(\lambda^*) = 0$$

Then it follows that:

$$G(\lambda^*) + \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m} F_{\theta,m,N}(\lambda^*) = \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m} F_{\theta,m,N}(\lambda^*) - \left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m} F_{\theta,m}(\lambda^*)$$

and so:

$$\left|G(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} F_{\theta,m,N}(\lambda^*)\right| \le \left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*)$$

Since $G(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} F_{\theta,m,N}(\lambda^*)$ is given and, $\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*)$ has computable upper bound, we can define an enclosure for λ^*, by solving the following system of inequalities:

$$-\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*) \le G(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} F_{\theta,m,N}(\lambda^*) \le \left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*) \tag{3.20}$$

Its solution is an interval containing λ^* and over which the graph $G(\lambda^*) + \left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m} F_{\theta,m,N}(\lambda^*)$ is trapped between the graphs:

$$-\left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*) \text{ and } \left|\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right|^{-m} T_N(\lambda^*)$$

Use the fact that $F_{\theta,m,N}(\lambda) \to F_{\theta,m}(\lambda)$ converges uniformly over any compact set and since λ^* is a simple root, we obtain for large N:

$$\frac{\partial}{\partial\lambda}\left(G(\lambda) + \frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m} F_{\theta,m,N}(\lambda) \ne 0$$

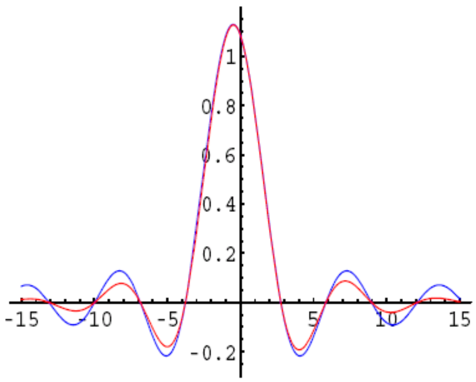

Fig. 1: $\Delta(\lambda)$, $\Delta_N(\lambda)$ with N = 40, m = 10 and θ = 1/15

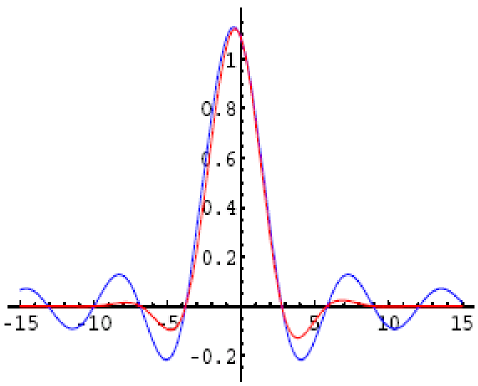

Fig. 2: $\Delta(\lambda)$, $\Delta_N(\lambda)$ with N = 40, m = 20 and θ = 1/10

in a neighborhood of λ^*. Hence the graph of $G(\lambda)+\left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}F_{\theta,m,N}(\lambda)$ intersects the graphs $-\left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m}T_N(\lambda)$ and $\left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m}T_N(\lambda)$ at two points with abscissae a-(λ^*, N)$\le$ $a_+(\lambda^*, N)$ and the solution of the system of inequalities (3.20) is the interval:

$$I_N(\lambda^*):=[a_-(\lambda^*,N),a_+(\lambda^*,N)]$$

and in particular $\lambda^*\in I_N(\lambda^*)$. Now, we summarize the above idea in the following lemma, (Boumenir, 2000a).

Lemma 5: For any eigenvalue λ^*:

- There exists N_0 such that $\lambda^*\in I_N(\lambda^*)$ for $N>N_0$
- $[a_-(\lambda^*, N), a_+(\lambda^*, N)]\to\{\lambda^*\}$ as $N\to\infty$

Proof: Since all eigenvalues of (2.1-2.3) are simple, then for N large enough we have

$$\frac{\partial}{\partial\lambda}\left(G(\lambda)+\left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}F_{\theta,m,N}(\lambda)\right)>0$$

say, in a neighborhood of λ^*. Now we choose N_0 such that:

$$G(\lambda)+\left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}F_{\theta,m,N}(\lambda)=\pm\left|\frac{\sin\theta\lambda}{\theta\lambda}\right|^{-m}T_{N0}(\lambda)$$

has two distinct solutions which we denote by $a_-(\lambda^*, N_0)\le a_+(\lambda^*, N_0)$. The decay of $T_N(\lambda)\to 0$ as $N\to\infty$ will ensure the existence of the solutions $a_-(\lambda^*, N)$ and $a_+(\lambda^*, N)$ as $N\to\infty$. For the second point we recall that $F_{\theta,m,N}(\lambda)\to F_{\theta,m}(\lambda)$ as $N\to\infty$. Hence by taking the limit we obtain:

$$G(a_+(\lambda^*,\infty))+\left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}F_{\theta,m}(a_+(\lambda^*,\infty))=0$$

$$G(a_-(\lambda^*,\infty))+\left(\frac{\sin\theta\lambda^*}{\theta\lambda^*}\right)^{-m}F_{\theta,m}(a_-(\lambda^*,\infty))=0$$

that is $\Delta(a_+) = \Delta(a_-) = 0$. This leads us to conclude that $a_+ = a_- = \lambda^*$, since λ^* is a simple root.

Examples: In this section, we now illustrate the above theory by looking at two simple examples where eigenvalue
enclosures are obtained. We also indicate the effect of the parameters m and by several choices. Both numerical results and the associated figures prove the credibility of the method. In the following examples, we consider $\lambda_{k,N}$ be the kth root of $G(\lambda)+\left(\frac{\sin\theta\lambda}{\theta\lambda}\right)^{-m}F_{\theta,m,N}(\lambda)=0$. Also, in the following examples, we observe that $\lambda_{k,N}$ and the exact solution λ are all inside the interval $[a_-, a_+]$.

Example 1: Consider the boundary value problem:

$$-y''(x,\lambda)+xy(x,\lambda)=\lambda(2iy'(x,\lambda)+y(x,\lambda)),\ 0\le x\le 1 \quad (4.1)$$

$$U_1(y):=y(0,\lambda)=0,\ U_2(y):=y(1,\lambda)=0 \quad (4.2)$$

This problem is a special case of problem (2.1-2.3) when $q(x) = x$, $\delta = \gamma = 0$. After some easy calculations:

$$G(\lambda):=\frac{\sin\sqrt{\lambda+\lambda^2}}{\sqrt{\lambda+\lambda^2}} \quad (4.3)$$

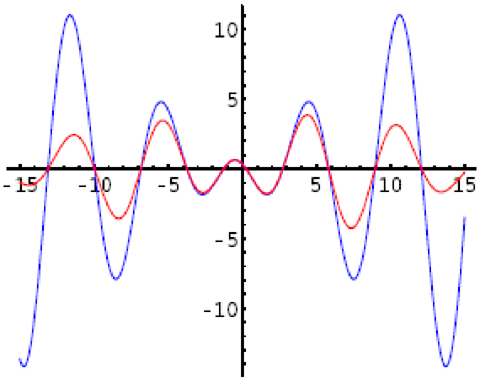

Fig. 3: $\Delta(\lambda)$, $\Delta_N(\lambda)$ with N = 30, m = 8 and θ = 1/11

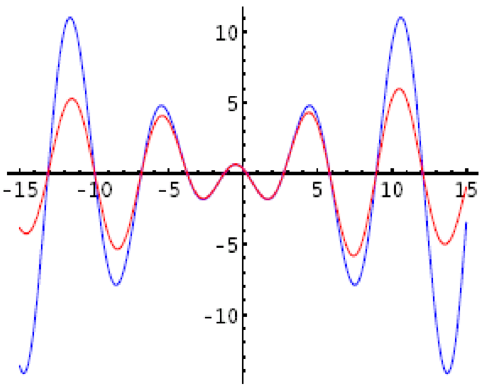

Fig. 4: $\Delta(\lambda)$, $\Delta_N(\lambda)$ with N = 30, m = 5 and θ = 2/25

$$N = 40, m = 10, \theta = \frac{1}{15}, E_9(F_{\theta,m}) = 3.71701 \times 10^{13}$$

$$N = 40, m = 20, \theta = \frac{1}{10}, E_{19}(F_{\theta,m}) = 2.19108 \times 10^{24}$$

Example 2: Consider the boundary value problem:

$$-y''(x,\lambda) + x^2 y(x,\lambda) = \lambda(2iy'(x,\lambda) + y(x,\lambda)), 0 \le x \le 1 \quad (4.4)$$

$$U_1(y) := y'(0,\lambda) + i\lambda y(0,\lambda) = 0, \\ U_2(y) := y'(1,\lambda) + i\lambda y(1,\lambda) = 0 \quad (4.5)$$

This problem is a special case of problem (2.1-2.3) when $q(x) = x^2$, $\delta = \gamma = \frac{\pi}{2}$. After some easy calculations:

$$G(\lambda) := -\sqrt{\lambda + \lambda^2} \sin \sqrt{\lambda + \lambda^2} \quad (4.6)$$

$$N = 30, m = 8, \theta = \frac{1}{11}, E_7(F_{\theta,m}) = 1.2251 \times 10^9$$

$$N = 30, m = 5, \theta = \frac{2}{25}, E_4(F_{\theta,m}) = 301631$$

CONCLUSION

In this study, we have used the regularized sampling method introduced recently (Chadan and Sabatier, 1989) to compute the eigenvalues of second-order operator pencil of the form Q-λP, where Q is second order self adjoint differential operator and P is a first order and $\lambda \in C$ is an eigenvalue parameter. We recall that this method constitutes an improvement upon the method based on Shannon's sampling theory introduced in (Boumenir, 1999) since it uses a regularization avoiding any multiple integration. The method allows us to get higher order estimates of the eigenvalues at a very low cost. We have presented two examples to illustrate the method and compared the computed eigenvalues with the exact ones when they are available. In these examples we observed, in Tables 1-4, that $\lambda_{k,N}$ and the exact solution λ_k are all inside the enclosure interval $[a_-, a_+]$, and also we illustrated, in Fig. 1-4, a slight different between $\Delta(\lambda)$ and $\Delta_N(\lambda)$ for different values of N, m and θ.

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