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Quadratic Approximation for Singular Integrals

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Abstract: In this study we present a new approach based on a quadratic approximation for singular integrals of Cauchy type, by using a small technical we arrive to eliminate completely the singularity of this integral. Noting that, this approximation is destined to solve numerically all singular integral equations with Cauchy kernel type on an oriented smooth contour.

Key words: Quadratic approximation, singular integral equations, oriented smooth contour, eliminate completely, quadratic method, singular integral operator

INTRODUCTION

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to singular integral equations with Cauchy kernel type: Eq. 1:

$$a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt + \int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0) \qquad (1)$$

where, Γ designates an oriented smooth contour, the points t and t_0 are on Γ.

The Eq. 1 plays an important role in modern numerical computations in the applied sciences, in particular in the applied mathematical.

Our schemes describe the quadratic method for the approximation of singular integral operator with Cauchy kernel Eq. 2:

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt, \quad t, t_0 \in \Gamma \qquad (2)$$

by a sequence of numerical integration operators.

Noting that, for the existence of the principal value of the integral (2) for a given density $\varphi(t)$ we will need more than mere continuity. In other words, the density $\varphi(t)$ has to satisfy the Hölder condition $H(\mu)$ (Muskhelishvili, 2008).

The function $\varphi(t)$ will be said to satisfy a Hölder condition on Γ if for any two points t_1 and t_2 of Γ:

$$|\varphi(t_2) - \varphi(t_1)| \le A|t_2 - t_1|^\mu, \quad 0 < \mu \le 1$$

where, A is positive constant, called the Holder constant and μ the Holder index.

The quadrature: Noting by t the parametric complex function t(s) of the curve Γ defined by:

$$t(s) = x(s) + iy(s), \quad a \le s \le b$$

where, x(s) and y(s) are continuous functions on the finite interval of definition [a, b] and have a continuous first derivatives x'(s) and y'(s) never simultaneously null. Let N be an arbitrary natural number, generally we take it large enough and divide the interval [a, b] into N equal subintervals of [a, b]:

$$[a,b] = \{a = s_0 < s_1 < \ldots\ldots < s_N = b\}$$

be called I_1 to I_N so that, we have $I_{\sigma+1}$ $[s_\sigma, s_{\sigma+1}]$:

$$s_\sigma = a + \sigma\frac{l}{N}, \; l = b - a, \; \sigma = 0,1,2,\ldots\ldots, N$$

Further, fixing a natural number M>1 and divide each of segments $[s_\sigma, s_{\sigma+1}]$ by the equidistant points:

$$s_{\sigma k} = s_\sigma + k\frac{h}{2M}, \quad h = \frac{l}{N}, \quad k = 0,1,\ldots, 2M$$

In other words, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision:

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < \ldots\ldots < s_{\sigma 2M} = s_{\sigma+1}\}$$

Denoting by:

$$t_\sigma = t(s_\sigma), \; t_{\sigma k} = t(s_{\sigma k}); \quad \sigma = 0,1,2,\ldots, N; \quad k = 0,1,\ldots\ldots, 2M$$

Assuming that for the indices σ,v = 0,1,2,…N-1 the points t and t_0 belong respectively to the arcs $\overset{\frown}{t_\sigma t_{\sigma+1}}$

and $\widehat{t_v t_{v+1}}$ where $\widehat{t_\alpha t_{\alpha+1}}$ designates the smallest arc with ends t_a and t_{a+1} (Nadir, 1985; 1998; Nadir and Antidze, 2004; Nadir and Lakehali, 2007; Sanikidze, 1970; Antidze, 1975).

For an arbitrary number σ = 0,1,2,…N-1 we define the spline function $S_2(\varphi,t,\sigma)$ dependents of φ, t and σ which represents the quadratic approximation of the function density φ(t) on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. As we know, the interval $[t_\sigma, t_{\sigma+1}]$ is divided into a subintervals $[t_{\sigma k}, t_{\sigma(k+2)}]$ of length $(t_{\sigma(k+2)}-t_{\sigma k})$ k = 2i, I = 0,1,...,M-1. We interpolate the function density φ(t) with respect to the values $\varphi(t_{\sigma k})$, $\varphi(t_{\sigma(k+1)})$ and $\varphi(t_{\sigma(k+2)})$ at the points $t_{\sigma k}$, $t_{\sigma(k+1)}$ and $t_{\sigma(k+2)}$ respectively with a quadratic polynomial, given by the following formula.

For $t_{ok} \le t \le_{\sigma(k+2)}$ Eq. 3:

$$S_2(\varphi;t,\sigma) = \frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k}) - \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) + \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)}) \quad (3)$$

This spline function exists and is unique also called a quadratic interpolating polynomial.

Define for an arbitrary numbers σ and v such that 0≤σ,v≤N-1 the function $\beta_{\sigma v}(\varphi,t,t_0)$ dependents of φ,t and t_0 by Eq. 4:

$$\beta_{\sigma v}(\varphi,t,t_0) = S_2(\varphi,t,\sigma) - S_2(\varphi,t_0,v) \quad (4)$$

where, $t\in[t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_v,t_{v+1}]$ and noting by $\Psi_{\sigma v}(\varphi,t,t_0)$ the quadratic approximation of the density φ(t) at the point $t\in[t_\sigma,t_{\sigma+1}]$ Eq. 5:

$$\psi_{\sigma v}(\varphi,t,t_0) = \varphi(t_0) + \beta_{\sigma v}(\varphi,t,t_0) \quad (5)$$

Replacing the density φ(t) by this last quadratic approximation $\Psi_{\sigma v}(\varphi, t, t_0)$ in the singular integral (2):

$$F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt$$

In order to obtain the following approximation Eq. 6:

$$S(\varphi,t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\psi_{\sigma v}(\varphi,t,t_0)}{t-t_0} dt = \varphi(t_0) + \frac{1}{\pi i}\int_\Gamma \frac{\beta_{\sigma v}(\varphi,t,t_0)}{t-t_0}dt \quad (6)$$

Theorem: Let Γ be an oriented smooth contour and let φ be a density function defined on Γ and satisfies the Hölder condition H(μ) then, the following estimation:

$$|F(t_0) - S(\varphi,t_0)| \le \max\left(\frac{C\ln(2MN)}{(2MN)^\mu}, \frac{C}{N^\mu}\right); \quad N, M > 1$$

Holds for all t and t_0 on the contour Γ. The constant C depends only of Γ.

Proof: Choosing the points $t\in[t_\sigma,t_{\sigma+1}]$ and $t_0\in[t_v,t_{v+1}]$ then for $t_{\sigma k} \le t \le t_{\sigma(k+2)}$ and $t_{ik} \le t_0 \le t_{v(k+2)}$ we have Eq. 7:

$$\begin{aligned}\varphi - \psi_{\sigma v}(\varphi,t,t_0) &= \varphi(t) - \varphi(t_0) \\ &-\Big\{\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k}) \\ &- \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) \\ &+ \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)}) \\ &- \frac{(t_0-t_{v(k+1)})(t_0-t_{v(k+2)})}{(t_{v(k+1)}-t_{vk})(t_{v(k+2)}-t_{vk})}\varphi(t_{vk}) \\ &+ \frac{(t_0-t_{vk})(t_0-t_{v(k+2)})}{(t_{v(k+1)}-t_{vk})(t_{v(k+2)}-t_{v(k+1)})}\varphi(t_{v(k+1)}) \\ &- \frac{(t_0-t_{vk})(t_0-t_{v(k+1)})}{(t_{v(k+2)}-t_{vk})(t_{v(k+2)}-t_{v(k+1)})}\varphi(t_{v(k+2)})\Big\}\end{aligned} \quad (7)$$

Taking into account the expression (7) we get Eq. 8:

$$\frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)-\psi_{\sigma v}(\varphi,t,t_0)}{t-t_0}dt = \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}} \frac{\varphi(t)-\psi_{\sigma v}(\varphi,t,t_0)}{t-t_0}dt \quad (8)$$

Hence:

$$\begin{aligned}F(t_0) - S(\varphi,t_0) &= \frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}} \frac{\varphi(t)-\varphi(t_0)}{t-t_0} \\ &-\Big\{\frac{(t-t_{\sigma(k+1)})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k}) \\ &- \frac{(t-t_{\sigma k})(t-t_{\sigma(k+2)})}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) \\ &+ \frac{(t-t_{\sigma k})(t-t_{\sigma(k+1)})}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)}) \\ &- \frac{(t_0-t_{v(k+1)})(t_0-t_{v(k+2)})}{(t_{v(k+1)}-t_{vk})(t_{v(k+2)}-t_{vk})}\varphi(t_{vk}) \\ &+ \frac{(t_0-t_{vk})(t_0-t_{v(k+2)})}{(t_{v(k+1)}-t_{vk})(t_{v(k+2)}-t_{v(k+1)})}\varphi(t_{v(k+1)}) \\ &- \frac{(t_0-t_{vk})(t_0-t_{v(k+1)})}{(t_{v(k+2)}-t_{vk})(t_{v(k+2)}-t_{v(k+1)})}\varphi(t_{v(k+2)})\Big\}\frac{1}{t-t_0}dt\end{aligned}$$

It can be seen that, the equality $t-t_0 = 0$ is possible only when σ = v-1,v+1 and v For the two first cases the integral (8) exists when t tends to t_0. The last case, if $\sigma = v$ we can easily seeing that, the function $\beta_{\sigma\sigma}(\varphi, t, t_0)$ contains $(t-t_0)$ as factor this means, for the points $t, t_0 \in [t_\sigma, t_{\sigma+1}]$ we write Eq. 9:

$$\beta_{\sigma\sigma}(\varphi, t, t_0) = S_2(\varphi, t, \sigma) - S_2(\varphi, t_0, \sigma)$$

Hence, for $t_{\sigma k} \le t, t_0 \le t_{\sigma(k+2)}$:

$$\begin{aligned}\beta_{\sigma\sigma}(\varphi, t, t_0) &= -\{\frac{(t-t_0)[(t-t_{\sigma(k+1)})+(t_0-t_{\sigma(k+2)})]}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma k})}\varphi(t_{\sigma k}) \\ &\quad -\frac{(t-t_0)[(t-t_{\sigma(k+2)})+(t_0-t_{\sigma k})]}{(t_{\sigma(k+1)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+1)}) \\ &\quad +\frac{(t-t_0)[(t-t_{\sigma k})+(t_0-t_{\sigma(k+1)})]}{(t_{\sigma(k+2)}-t_{\sigma k})(t_{\sigma(k+2)}-t_{\sigma(k+1)})}\varphi(t_{\sigma(k+2)})\}\end{aligned} \quad (9)$$

Passing now to the estimation of the expression (8):

For $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_v, t_{v+1}]$ with the conditions $\sigma \ne v-1, v+1$ and v we obtain:

$$\left| \begin{aligned} &1/(\pi i)\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}t_{\sigma(2k+2)}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0} \\ &-\{\frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\varphi(t_{\sigma 2k}) \\ &-\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+1)}) \\ &+\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+2)}) \\ &-\frac{(t_0-t_{v(2k+1)})(t_0-t_{v(2k+2)})}{(t_{v(2k+1)}-t_{v2k})(t_{v(2k+2)}-t_{v2k})}\varphi(t_{v2k}) \\ &+\frac{(t_0-t_{v2k})(t_0-t_{v(2k+2)})}{(t_{v(2k+1)}-t_{v2k})(t_{v(2k+2)}-t_{v(2k+1)})}\varphi(t_{v(2k+1)}) \\ &-\frac{(t_0-t_{v2k})(t_0-t_{v(2k+1)})}{(t_{v(2k+2)}-t_{v2k})(t_{v(2k+2)}-t_{v(2k+1)})} \\ &\varphi(t_{v(2k+2)})\}\frac{1}{t-t_0}dt \end{aligned} \right| = O(\frac{\ln 2MN}{(2M)^\mu N^\mu})$$

Indeed, it is clear that:

$$\max_{t_0 \in t_v t_{v+1}} |\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}}^{t_{\sigma(2k+2)}}\frac{\varphi(t)-\phi(t_0)}{t-t_0}dt| = O(\frac{\ln 2MN}{(2M)^\mu N^\mu})$$

And:

$$\left| \begin{aligned} &\frac{1}{\pi i}\sum_{\sigma=0}^{N-1}\sum_{k=0}^{M-1}\int_{t_{\sigma 2k}}^{t_{\sigma(2k+2)}} \\ &-\{\frac{(t-t_{\sigma(2k+1)})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma 2k})}\varphi(t_{\sigma 2k}) \\ &-\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+2)})}{(t_{\sigma(2k+1)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+1)}) \\ &+\frac{(t-t_{\sigma 2k})(t-t_{\sigma(2k+1)})}{(t_{\sigma(2k+2)}-t_{\sigma 2k})(t_{\sigma(2k+2)}-t_{\sigma(2k+1)})}\varphi(t_{\sigma(2k+2)}) \\ &-\frac{(t_0-t_{v(2k+1)})(t_0-t_{v(2k+2)})}{(t_{v(2k+1)}-t_{v2k})(t_{v(2k+2)}-t_{v2k})}\varphi(t_{v2k}). \\ &+\frac{(t_0-t_{v2k})(t_0-t_{v(2k+2)})}{(t_{v(2k+1)}-t_{v2k})(t_{v(2k+2)}-t_{v(2k+1)})}\varphi(t_{v(2k+1)}) \\ &-\frac{(t_0-t_{v2k})(t_0-t_{v(2k+1)})}{(t_{v(2k+2)}-t_{v2k})(t_{v(2k+2)}-t_{v(2k+1)})} \\ &\varphi(t_{v(2k+2)})\}\frac{1}{t-t_0}dt \end{aligned} \right| = O(\frac{\ln 2MN}{(2M)^\mu N^\mu})$$

Further, for the three cases mentioned above $\sigma = v-1, v+1$ and v using the smoothness of the contour Γ and the function $\varphi(t)$ in the Hölder space $H(\mu)$ we get:

$$|\int_{t_v t_{v+1}}\frac{\varphi(t)-\varphi(t_0)}{t-t_0}dt| \le A\int_{s_v}^{s_{v+1}}|s-s_0|^{\mu-1}ds = O(N^{-\mu})$$

Example 1: Consider the singular integral:

$$I = F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt$$

where, Γ designates the circle centered at the origin point 0 with a unit radius and the density function φ is given by the following expression:

$$\varphi(t)=1/(t+2)$$

Table 1: Singular integral with $\varphi(t)=1/(t+2)$

N	M	$\|\|I-I_p\|\|_1$	$\|\|I-I_p\|\|_2$	$\|\|I-I_3\|\|_\infty$
10	3	4.484951E-04	2.2449576E-04	1.1798739E-04
10	4	2.5682151E-05	1.2854857E-05	6.8843365E-06
10	5	1.1920929E-06	7.300048E-07	5.9604645E-07

Example 2: We take the singular integral:

$$I = F(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t)}{t-t_0}dt$$

where, Γ designates the circle centered at the origin point 0 with a unit radius and the density function φ is given by the following expression:

$$\varphi(t) = \frac{t+1}{t^2+5t+6}$$

Table 2: Singular integral with $\varphi(t) = \frac{t+1}{t^2+5t+6}$

N	M	$\|I-I_p\|_1$	$\|I-I_p\|_2$	$\|I-I_3\|_\infty$
10	3	3.6347099E-04	2.1240809E-04	1.7364323E-04
10	4	1.2820773E-04	7.9004079E-05	6.6116452E-05
10	5	6.1048195E-05	3.8276499E-05	3.1117350E-05

Numerical experiments: Using our approximation, we apply the algorithms to singular integrals and we present results concerning the accuracy of the calculations, in these numerical experiments each Table 1 and 2 represent the exact principal value of the singular integral and I_p corresponds to the approximate calculation produced by our approximation at interpolation point's values.

CONCLUSION

This approximation can be used to remove integrable singularity. It was tested for the numerical calculus of many singular integrals of Cauchy type, where it gave good results.

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