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Unique Representation of Positive Integers as a Sum of Distinct Tribonacci Numbers

Salim Badidja and Abdelmadjid Boudaoud

Laboratory of Pure and Applied Mathematics (L.M.P.A.),
Faculty of Mathematics and Computer Sciences, University of Mohamed BOUDIAF of M'sila, M'sila, Algeria

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Corresponding Author:
Salim Badidja
Laboratory of Pure and Applied Mathematics (L.M.P.A.), Faculty of Mathematics and Computer Sciences, University of Mohamed BOUDIAF of M'sila, M'sila, Algeria
Email: bsalimmah@gmail.com

Abstract: Let $(T_m)_{m\geq 1}$ be the tribonacci sequence. We show that every integer $N \geq 1$ can be written as a sum of the terms $\alpha_m T_m$, where m runs over the set of strictly positive integers and α_m $(m \geq 1)$ are either 1 or 0. The previous representation of N is unique if each time that we have $\alpha_m = 1$ then at least the two coefficients directly following α_m are zero, i.e., $\alpha_{m+1} = \alpha_{m+2} = 0$.

Keywords: Linear Recurrent Sequences, Tribonacci Numbers, Representation of Integers

Introduction

The tribonacci numbers $\{T_n\}_1^\infty$, are defined by:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 4$$

where, $T_1 = T_2 = 1$, $T_3 = 2$ (Koshy, 2001). The first tribonacci numbers are 1, 1, 2, 4, 7, 13, 24,... .

Recall that in 1969 J. L Brown represents the integers by a sum of distinct Lucas numbers (Brown, Jr, 1969), also in 1986 Jukka Puhko treated the Fibonacci and Lucas representation and theorem of Lekkerkerker (Pihko, 1986), in the mathematical literature there are several references concerning the representation of a positive integer by a finite sum of elements of an infinite sequence of integers. Our main goal is to prove that every positive integer $N > 0$ can be represented by $\sum_1^\infty \alpha_m T_m$, where α_m $(m \geq 1)$ are either 1 or 0. This representation is unique if the coefficients α_m $(m \geq 1)$ satisfy:

$$\begin{cases} \alpha_m \alpha_{m+1} = 0 \\ \alpha_m \alpha_{m+2} = 0, \end{cases} \quad for\ m \geq 1$$

where, taking into account $T_1 = T_2 = 1$, the uniqueness is in the following sense.

Definition 1.1

Two representations $\sum_{m=1}^\infty \alpha_m T_m$ and $\sum_{m=1}^\infty \beta_m T_m$ are called identical if:

- $\alpha_m = \beta_m$ for $m \geq 1$ or
- $\alpha_m = \beta_m$ for $m \geq 3$ and $(\alpha_1, \beta_1) = (1, 0)$ and $(\alpha_2, \beta_2) = (0, 1)$ or $(\alpha_1, \beta_1) = (0, 1)$ and $(\alpha_2, \beta_2) = (1, 0)$

Main Results

Theorem 2.1

For every integer $N > 0$, there exists an integer $i_1 > 0$ such that $N = \sum_{i=1}^{i_1} \alpha_i T_i$, where $\alpha_i \in \{0, 1\}$.

Proof

Let $N > 0$ be an integer. Firstly, we show that the sequence $(T_i)_{i\geq 1}$ satisfies:

$$T_{i+1} \leq 2T_i, \forall i \geq 1$$

Indeed, we easily check this for $i = 1, 2, 3$. For $i \geq 4$, we have by definition:

$$T_{i+1} = T_i + T_{i-1} + T_{i-2} \leq T_i + \left(T_{i-1} + T_{i-2} + T_{i-3}\right) = 2T_i$$

Now, putting $\Delta_0 = N$. Let i_1 be the largest integer such that:

$$T_{i_1} \le \Delta_0 < T_{i_1+1}$$

Putting $\Delta_1 = \Delta_0 - T_{i_1}$. If $\Delta_1 = 0$, then $\Delta_0 = T_{i_1}$ i.e., $N = T_{i_1}$ and this completes the proof. Otherwise, $\Delta_1 > 0$ and in this case necessarily $\Delta_1 = \Delta_0 - T_{i_1} < T_{i_1}$ because if not then $\Delta_0 - T_{i_1} \ge T_{i_1}$ yields $\Delta_0 \ge 2T_{i_1} \ge T_{i_1+1}$ but this contradicts the choice of i_1 which yields $T_{i_1} \le \Delta_0 < T_{i_1+1}$. Now we choose $i_2 < i_1$ such that:

$$T_{i_2} \le \Delta_1 < T_{i_2+1}$$

Putting $\Delta_2 = \Delta_1 - T_{i_2}$. If $\Delta_2 = 0$, then $\Delta_1 - T_{i_2} = 0$, or $\Delta_1 = T_{i_2}$ and consequently $\Delta_1 = T_{i_2} = N - T_{i_1}$ since $\Delta_0 = N$. Hence, $N = T_{i_1} + T_{i_2}$ and this ends the proof. Otherwise, $\Delta_2 > 0$ and in this case necessarily $\Delta_2 = \Delta_1 - T_{i_2} < T_{i_2}$ if not then $\Delta_1 - T_{i_2} \ge T_{i_2}$ which signifies that $\Delta_1 \ge 2T_{i_2} \ge T_{i_2+1}$ but this contradicts the fact that $T_{i_2} \le \Delta_1 < T_{i_2+1}$.

Continuing in this process, we obtain a decreasing sequence $i_1 > i_2 > i_3 > ... > ...$ which should stabilize at some i_k. Hence $N = T_{i_1} + T_{i_2} + ... + T_{i_k}$ and consequently $N = \sum_{i=1}^{i_1} \alpha_i T_i$, where $\alpha_i \in \{0, 1\}$ for $(1 \le i \le i_1)$.

Proposition 2.1

For every integer $n \ge 5$, we have:

$$T_n = 3 + T_{n-2} + 2\sum_{i=2}^{n-3} T_i$$

Proof

From the definition of $(T_n)_{n\ge1}$, easily we have:

$$T_n = T_1 + T_3 + T_{n-2} + 2\sum_{i=2}^{n-3} T_i \ , n \ge 5$$

Or:

$$T_n = 3 + T_{n-2} + 2\sum_{i=2}^{n-3} T_i \ , n \ge 5$$

Theorem 2.2

Let $N \ge 1$ be an integer represented by $\sum_{m=1}^{\infty} \alpha_m T_m$, for $m \ge 1$:

$$\alpha_m \in \{0,1\} \ and \begin{cases} \alpha_m \alpha_{m+1} = 0 \\ \alpha_m \alpha_{m+2} = 0 \end{cases} .$$

Then this representation is unique.

Proof

Before the proof we need the following lemma.

Lemma 2.1

Under the same assumption of theorem 2.2 and for $k \ge 6$, we have:

$$\sum_{m=1}^{k} \alpha_m T_m < 2\sum_{m=2}^{k-2} T_m + T_{k-2}$$

Proof

Firstly we prove by induction that:

$$2T_{k-2} - T_{k-1} - 1 \ge 0, k \ge 6$$

This formula is satisfied for $k = 6$, because:

$$\begin{aligned} 2T_4 - T_5 - 1 &= 8 - 7 - 1 \\ &= 0 \ge 0 \end{aligned}$$

Suppose that:

$$2T_{k-2} - T_{k-1} - 1 \ge 0, k \ge 6$$

and prove that:

$$2T_{(k+1)-2} - T_{(k+1)-1} - 1 = 2T_{k-1} - T_k - 1 \ge 0$$

In fact:

$$\begin{aligned} &2T_{k-1} - T_k - 1 = T_{k-1} - T_{k-2} - T_{k-3} - 1 \\ &\left(T_{k-2} + T_{k-3} + T_{k-4}\right) - T_{k-2} - T_{k-3} - 1 \\ &= T_{k-4} - 1 \end{aligned}$$

Since $T_2 - 1 = 0 \ge 0$ and the fact that $(T_i)_{i\ge4}$ is strictly increasing, we have always:

$$2T_{k-1} - T_k - 1 \ge 0, k \ge 6$$

This completes the proof.

Returning to the proof of the lemma. Putting:

$$\begin{aligned} L &= 2\sum_{m=2}^{k-2} T_m - \sum_{m=1}^{k} \alpha_m T_m + T_{k-2} \\ &= \sum_{m=2}^{k-2} \left(2 - \alpha_m\right) T_m - \alpha_1 - \alpha_{k-1} T_{k-1} - \alpha_k T_k + T_{k-2} \end{aligned}$$

and prove that $L > 0$. We distinguish two cases:

A) For $\alpha_k = 0$, we have:

$$L = \sum_{m=2}^{k-3}\left(2-\alpha_m\right)T_m - \alpha_1 - \alpha_{k-1}T_{k-1} + T_{k-2} + \left(2-\alpha_{k-2}\right)T_{k-2}$$

then:

$$L \geq \sum_{m=2}^{k-3}\left(2-\alpha_m\right)T_m + 2T_{k-2} - T_{k-1} - 1$$

Since:

$$\sum_{m=2}^{k-3}\left(2-\alpha_m\right)T_m > 0, k \geq 6$$

and from above:

$$2T_{k-2} - T_{k-1} - 1 \geq 0$$

Consequently $L > 0$.

B) $\alpha_k = 1$. Which implies in this case that $\alpha_{k-1} = \alpha_{k-2} = 0$. So:

$$\begin{aligned}
L &= \sum_{m=2}^{k-2}\left(2-\alpha_m\right)T_m - \alpha_1 - T_k + T_{k-2} \\
&= \sum_{m=2}^{k-3}\left(2-\alpha_m\right)T_m + \left(2-\alpha_{k-2}\right)T_{k-2} - \alpha_1 - T_k + T_{k-2} \\
&= \sum_{m=2}^{k-3}\left(2-\alpha_m\right)T_m + 3T_{k-2} - T_k - \alpha_1 \\
&= \sum_{m=2}^{k-4}\left(2-\alpha_m\right)T_m + \left(2-\alpha_{k-3}\right)T_{k-3} + 3T_{k-2} - T_k - \alpha_1 \\
&\geq \sum_{m=2}^{k-4}\left(2-\alpha_m\right)T_m + T_{k-3} + 3T_{k-2} - T_k - 1
\end{aligned}$$

From:

$$T_k = T_{k-1} + T_{k-2} + T_{k-3}$$

we get:

$$L \geq \sum_{m=2}^{k-4}\left(2-\alpha_m\right)T_m + 2T_{k-2} - T_{k-1} - 1$$

Since:

$$2T_{k-2} - T_{k-1} - 1 \geq 0$$

and:

$$\sum_{m=2}^{k-4}\left(2-\alpha_m\right)T_m > 0$$

then:

$$L > 0$$

Consequently:

$$\sum_{m=1}^{k}\alpha_m T_m < 2\sum_{m=2}^{k-2}T_m + T_{k-2}, \textit{for every } k \geq 6$$

Which completes the proof of lemma.

Let us return to the proof of theorem 2.2 by taking $N \geq 1$ and assuming that N has two non identical representations

$$N = \sum_{m=1}^{\infty}\alpha_m T_m = \sum_{m=1}^{\infty}\beta_m T_m$$

where:

$$\begin{cases}\alpha_m\alpha_{m+1} = 0 \\ \alpha_m\alpha_{m+2} = 0\end{cases}, \textit{for } m \geq 1$$

and the same constraints for β_m ($m \geq 1$).

That is, $\sum_{1}^{\infty}|\alpha_m - \beta_m| \neq 0$. Let k be the largest value of m such that $\alpha_k \neq \beta_k$, we may assume without loss of generality that $\alpha_k \neq 1$ and $\beta_k = 0$. Since the validity of lemma 2.1 and proposition 2.1 is for $k \geq 6$, we distinguish in the sequel two cases.

A) $1 \leq k \leq 6$.

k = 1

This means that $\alpha_1 = 1$, $\beta_1 = 0$ and $\alpha_m = \beta_m$, for $m \geq 2$.

Since:

$$\sum_{m=2}^{+\infty}\alpha_m T_m = \sum_{m=2}^{+\infty}\beta_m T_m$$

and:

$$\begin{aligned}
&\sum_{m=1}^{1}\alpha_m T_m + \sum_{m=2}^{+\infty}\alpha_m T_m = 1 + \sum_{m=2}^{+\infty}\alpha_m T_m \neq \sum_{m=1}^{+\infty}\beta_m T_m \\
&+\sum_{m=2}^{+\infty}\beta_m T_m = 0 + \sum_{m=2}^{+\infty}\beta_m T_m
\end{aligned}$$

there is not two non identical representations, because N cannot be represented by two different values.

k = 2

Which yields $\alpha_2 = 1$, $\beta_2 = 0$ and $\alpha_m = \beta_m$ for $m \geq 3$.

With these coefficients we can represent N as:

$$N = \sum_{m=1}^{+\infty} \alpha_m T_m = \sum_{m=1}^{+\infty} \beta_m T_m$$

if and only if $\alpha_1 = 0$ and $\beta_1 = 1$. Thus:

$$N = 0.T_1 + 1.T_2 + \sum_{m=3}^{+\infty} \alpha_m T_m = 1.T_1 + 0.T_2 + \sum_{m=3}^{+\infty} \beta_m T_m$$

but a such representations are identical in the sense of the definition 1.1. If $\beta_1 = 0$, then:

$$\sum_{m=1}^{+\infty} \alpha_m T_m \neq \sum_{m=1}^{+\infty} \beta_m T_m$$

and there are not two non identical representations, since N can not be represented by two different values.

k = 3.

That is, $\alpha_3 = 1$, $\beta_3 = 0$ and for $m \geq 4$, $\alpha_m = \beta_m$. In this case the greatest value of $\sum_{m=1}^{3} \beta_m T_m$ is 1 and in contrast $\sum_{m=1}^{3} \alpha_m T_m = 2$. Hence:

$$\sum_{m=1}^{3} \alpha_m T_m + \sum_{m=4}^{+\infty} \alpha_m T_m = 2 + \sum_{m=4}^{+\infty} \alpha_m T_m \neq \sum_{m=1}^{3} \beta_m T_m + \sum_{m=4}^{+\infty} \beta_m T_m$$

Then, as above, there are not two non identical representations, since N can not be represented by two different values.

k = 4

That is, $\alpha_4 = 1$, $\beta_4 = 0$ and for $m \geq 5$, $\alpha_m = \beta_m$. In this case the greatest value of $\sum_{m=1}^{4} \beta_m T_m$ is 2 and in contrast the smallest value of $\sum_{m=1}^{4} \alpha_m T_m$ is 4. Hence, N can not be represented by two different values, that is, there is not two non identical representations.

k = 5

That is, $\alpha_5 = 1$, $\beta_5 = 0$ and for $m \geq 6$, $\alpha_m = \beta_m$. In this case the greatest value of $\sum_{m=1}^{5} \beta_m T_m$ is 5 and in contrast the smallest value of $\sum_{m=1}^{5} \alpha_m T_m$ is 7. Thus, as above there is not two non identical representations.

k = 6

That is, $\alpha_6 = 1$, $\beta_6 = 0$ and for $m \geq 7$, $\alpha_m = \beta_m$. In this case the greatest value of $\sum_{m=1}^{6} \beta_m T_m$ is 8 and in contrast the smallest value of $\sum_{m=1}^{6} \alpha_m T_m$ is 13. Thus, as in the previous cases N can not be represented by two different values, that is, there is not two non identical representations.

B) $7 \leq k$. From the fact that $\alpha_m = \beta_m$ for $m \geq k + 1$ and:

$$N = \sum_{m=1}^{+\infty} \alpha_m T_m = \sum_{m=1}^{+\infty} \beta_m T_m$$

we have:

$$\sum_{m=1}^{k} \alpha_m T_m = \sum_{m=1}^{k} \beta_m T_m$$

Putting:

$$l = \sum_{m=1}^{k} \alpha_m T_m = \sum_{m=1}^{k} \beta_m T_m$$

Then:

$$l = \sum_{m=1}^{k} \alpha_m T_m = \sum_{m=1}^{k-1} \beta_m T_m$$

since $\beta_k = 0$. On the other hand $l \leq N$ and we can write:

$$l = \sum_{m=1}^{k} \alpha_m T_m = \sum_{m=1}^{k-1} \alpha_m T_m + T_k \geq T_k$$

According to lemma 2.1. we have:

$$l = \sum_{m=1}^{k-1} \beta_m T_m < 2\sum_{m=2}^{k-3} T_m + T_{k-3}$$

And by proposition 2.1, we have:

$$T_k - 3 - T_{k-2} = 2\sum_{m=2}^{k-3} T_m$$

So:

$$l < T_k - 3 - T_{k-2} + T_{k-3}$$

By the recurrent relation of T_n, we have:

$$l < T_{k-1} + T_{k-2} + T_{k-3} - 3 - T_{k-2} + T_{k-3}$$

That is:

$$l < T_{k-1} - 3 + 2T_{k-3}$$

Now to get $l < T_k$, we will prove that:

$$T_{k-1} - 3 + 2T_{k-3} < T_k$$

Indeed:

$$\begin{aligned}
&T_k - T_{k-1} + 3 - 2T_{k-3} = T_{k-1} + T_{k-2} + T_{k-3} - T_{k-1} + 3 - 2T_{k-3} \\
&= T_{k-2} + 3 - T_{k-3} \\
&= T_{k-3} + T_{k-4} + T_{k-5} + 3 - T_{k-3} \\
&= T_{k-4} + T_{k-5} + 3 > 0
\end{aligned}$$

Then:

$$l < T_{k-1} - 3 + 2T_{k-3} < T_k$$

Finally, we have:

$$T_k \le l < T_k$$

which is a contradiction. Hence the representation is unique.

Applications and Perspectives

The representation of a positive integer n by a sum of elements of a given sequence is an interesting problem which is well known in the mathematical literature; namely, unique representation of integers as sum of distinct Lucas numbers (Brown, Jr, 1969), Fibonacci and Lucas representation (Pihko, 1986), Cantor's development of a positive integer (Mercier, 2004),... . Our problem set in this context but with a well addition which is the uniqueness of representation.

As a perspective, the techniques used in our work can be employed in other problems for the same purposes. In this sense we can consider, for example, the case of the generalized tribonacci sequences and higher orders (Pentanacci, hexanacci, ...k-Fibonacci sequence..).

This field of mathematics which focuses on the study of words and formal languages combinatorics on words affects various areas of mathematics study, including algebra and computer science. Combinatorics of words is connected to many modern, as well as classical, fields of mathematics. Connections to combinatorics-actually being part of it - are obvious, but also connections to algebra are deep. Indeed, a natural environment of a word is a free semigroup.

More generally, the above connections can be illustrated as in Fig. 1 (Karhumaki, J.).

For more clarification we can take the sequences of words like the Fibonacci sequence of words on the binary alphabet {0, 1} can be de.ned by the recurrent relation:

$$\begin{cases} F_1 = 1, F_2 = 0 \\ F_n = F_{n-1}F_{n-2}\left(with\, the\, concatenation\, product\right) \end{cases}$$

$F_1 = 1$, $F_2 = 0$, $F_3 = 01$, $F_4 = 010$,.... . For further references on the subject see for example (Lotaire, 2002; Karhumaki and Berstel, 2003).

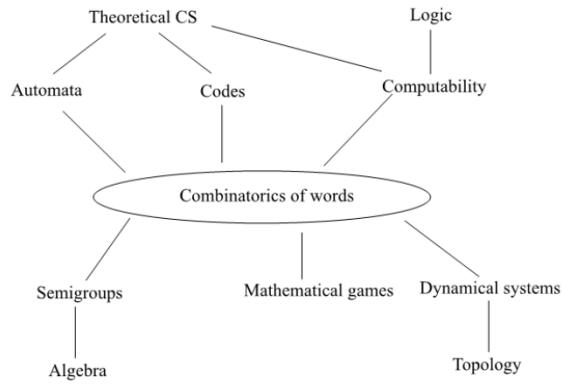

Fig. 1. Reference grid of the relationship between the fields of applications

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Author's Contributions

The authors worked together during the preparation of this article.

Ethics

The present work is really ours and there will be no ethical problem.

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