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Poisson Quasi-Maximum Likelihood Estimator-based CUSUM Test for Integer-Valued Time Series

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Abstract: This study considers the parameter change test for integer-valued time series models based on the Poisson quasi-maximum likelihood estimates. As a change point test, we consider the score vector-based CUSUM test and show that its limiting null distribution takes the form of a function of Brownian bridges. Moreover, the residual-based CUSUM tests are considered as alternatives. For evaluation, we conduct a Monte Carlo simulation study with Poisson, zero-inflated Poisson, negative binomial and Conway-Maxwell integer-valued generalized autoregressive conditional heteroscedastic models and Poisson integer-valued autoregressive models, and compare the performance of the proposed CUSUM tests. Our findings confirm that the proposed test is a functional tool for detecting a change point when the underlying distribution is unspecified.

Keywords: Time Series of Counts, INGARCH Model, INAR Model, Poisson QMLE, CUSUM Test

1. Introduction

Integer-valued time series have been intensively studied over the past decades in finance and economy, engineering and industry and environmental and health science. Since the seminal papers of McKenzie (1985), Alzaid and Al-Osh (1990), and Al-Osh and Aly (1992), integer-valued autoregressive (INAR) models with a binomial thinning operator have been popular among researchers. Later, integer-valued generalized autoregressive conditional heteroscedastic (INGARCH) models have emerged in the literature and attracted attention of researchers. Refer to Heinen (2003), Ferland *et al*. (2006), Fokianos *et al*. (2009), Neumann (2011), and Weiß (2018) for a general review of time series models of counts.

In both INAR and INGARCH models, the Poisson distribution has been widely used as the conditional distribution of present observations over past information. However, it fails to describe well the over-dispersion phenomenon frequently observed in real situations. To overcome this shortcoming, several authors have used other distributions, such as Negative Binomial (NB), zero-inflated generalized Poisson, and one-parameter exponential family distributions. We refer to Davis and Wu (2009), Zhu (2011, 2012), Christou and Fokianos (2014), and Davis and Liu (2016). In practice, however, approaches based on specific distributions can be problematic when the assumed distribution does not fit the dataset well. In this case, the Poisson Quasi-Maximum Likelihood Estimator (QMLE) method of Ahmad and Francq (2016) can be a good substitute to the MLE-based method. In particular, the Poisson QMLE method works well with INAR models because the conditional mean and variance in these models are unrelated to their underlying distributions. In this study, we demonstrate that the Poisson QMLE approach can be useful for applying the CUSUM method to integer-valued time series.

Since time series often undergo structural changes in their underlying models, the problem of detecting change points has been an important research topic in time series analysis. This subject has a long history and numerous articles exist in this literature stream. See Csörgo and Horváth (1997) and Lee *et al.* (2003) for a review of the CUSUM test. The change point test for integer-valued time series has been studied in both INAR and INGARCH models. Kang and Lee (2009, 2014), Fokianos and Fried (2010, 2012), Franke *et al*. (2012), Fokianos *et al*. (2014), and Lee *et al.* (2018). The CUSUM test compares the parameter estimates calculated from sequentially observed samples and detects a change when the CUSUM test statistic exceeds a predetermined value at some time point. Although the estimate-based CUSUM test performs satisfactorily in many situations, it often has severe size distortions when applied to GARCH-type models, as demonstrated by Kang and Lee (2014).

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Hence, the score vector-and residual-based tests have been used as alternatives. See Lee *et al*. (2004) and Lee and Lee (2019) for their background. Lee and Lee (2019) also showed that the CUSUM test based on standardized residuals can enhance the power markedly and Lee (2019) recently demonstrated that the residual-based CUSUM of squares test outperforms the Lee and Lee (2019) test for Poisson INGARCH(1,1) models to a great extent.

In this study, we compare the performance of Poisson QMLE-based CUSUM tests such as the score vector-based CUSUM test, the residual-based CUSUM test, and the residual-based CUSUM of squares test. Our simulation study reveals that the Poisson QMLE method is well incorporated with the CUSUM test. Our findings show that (i) the score vector-based CUSUM test is largely superior to the other tests in INGARCH models, (ii) only the residual-based CUSUM of squares test can detect a change of zero proportion, and (iii) the residual-based CUSUM test shows outstanding performance in dealing with INAR models. The remainder of this paper is organized as follows. Section 2 proposes the CUSUM test based on the score vectors with Poisson QMLE and derives the asymptotic result for this test. Section 3 reports the results of our simulation study using Poisson, zero-inflated Poisson, NB and Conway-Maxwell (COM) INGARCH(1,1) models as well as Poisson INAR(1) models. Section 4 provides concluding remarks.

2. Poisson QMLE-Based CUSUM Test

Let $\{Y_t,\ t \geq 1\}$ be a time series of counts with $\lambda_t = \lambda_t(\theta_0) = E(Y_t \mid \mathcal{F}_{t-1})$, where $\mathcal{F}_t$ is the σ-field generated by $Y_t, Y_{t-1},\ldots$ and θ_0 belongs to some parameter space $\theta \subset \mathbb{R}^d$. Further, we assume that $\lambda_t(\theta)$ is a function of random variables $Y_t, Y_{t-1},\ldots$, namely, $\lambda_t(\theta) = \lambda_t(Y_t, Y_{t-1},\ldots;\theta)$ and $\{Y_t, \lambda_t(\theta)\}$ is stationary and ergodic. An important example of $\{Y_t\}$ is the INGARCH model with the conditional distribution of the one-parameter exponential family in Davis and Liu (2016), namely:

$$Y_t \mid \mathcal{F}_{t-1} \sim p(y|\eta_t),$$
$$\lambda_t := E(Y_t|\mathcal{F}_{t-1}) = f_\theta(\lambda_{t-1}, Y_{t-1}), \tag{1}$$

where $\mathcal{F}_t$ is the σ-field generated by $\eta_1, Y_1,\ldots, Y_t, f_\theta(x, y)$ is a nonnegative bivariate function defined on $[0,\infty)\times\mathbb{N}_0(\mathbb{N}_0 = \mathbb{N}\cup\{0\})$ and $p(\cdot \mid \cdot)$ is a probability mass function given by

$$p(y \mid \eta) = \exp\{\eta y - A(\eta)\}h(y), y \geq 0,$$

where η is the natural parameter, $A(\eta)$ and $h(y)$ are known functions and $A'(\cdot)$ exists and is strictly increasing; further, $\eta_t = (A')^{-1}(\lambda_t)$. Y_t is stationary and ergodic if f satisfies the condition that for all $x, x' \geq 0$ and $y, y' \in \mathbb{N}_0$,

$$\sup_{\theta\in\Theta} | f_\theta(x,y) - f_\theta(x',y')| \leq \omega_1 \mid x - x' \mid + \omega_2 \mid y - y' \mid,$$

where $\omega_1,\ \omega_2 \geq 0$ satisfies $\omega_1 + \omega_2 < 1$. Model (1) accommodates a broad class of INGARCH models including Poisson and NB-INGARCH models. Our setup also includes INAR models, as shown by Ahmad and Francq (2016); see Example 2 in the next section.

Setting $\tilde{\lambda}_t := \tilde{\lambda}_t(\theta) = \lambda_t(Y_t,\ldots,Y_1, y_0,\ldots;\theta)$ for some fixed nonnegative integer y_0 (e.g., $y_0 = 0$), we obtain the Poisson QMLE of θ_0 by

$$\hat{\theta}_n = argmax_{\theta\in\Theta}\sum_{t=1}^{n}\tilde{\ell}_t(\theta),$$

where $\tilde{\ell}_t(\theta) = -\tilde{\lambda}_t(\theta) + Y_t \log\tilde{\lambda}_t(\theta)$. Here, when $\lambda_t = f_\theta(Y_{t-1}, \lambda_{t-1})$ as seen in Model (1), $\tilde{\lambda}_t$ are recursively obtained through the equation $\tilde{\lambda}_t = f_\theta(Y_{t-1}, \tilde{\lambda}_{t-1})$ with the preassigned initial values Y_0, λ_0.

Below are some regularity conditions, where $0<\rho<1$ and V represent a generic constant and integrable random variable, respectively, $\|\cdot\|$ denotes the L^1 norm for vectors and matrices and $E(\cdot)$ is taken under true parameter θ_0. In particular, the QMLE $\hat{\theta}_n$ is strongly consistent and asymptotic normal, as shown by Ahmad and Francq (2016):

(A1) θ_0 is an interior point that belongs to the compact parameter space $\Theta \in \mathbb{R}^d$.

(A2) $\theta \to \lambda_t(\theta)$ is continuous and for any t and $\theta \in \Theta, \lambda_t(\theta) \wedge \tilde{\lambda}_t(\theta) \geq c$ for some $c > 0$.

(A3) $\lambda_t(\theta) = \lambda_t(\theta_0)$ a.s. for some t if and only if $\theta = \theta_0$.

(A4) $EY_1^4 < \infty, E(\sup_{\theta\in\Theta}\lambda_1^4(\theta)) < \infty,$

$$E\left(\sup_{\theta\in\Theta}\left\|\frac{\partial\lambda_1(\theta)}{\partial\theta}\right\|^4 + \sup_{\theta\in\Theta}\left\|\frac{\partial^2\lambda_1(\theta)}{\partial\theta\partial\theta^T}\right\|^2\right) < \infty.$$

Moreover, $\frac{\partial^2\lambda_1(\theta)}{\partial\theta\partial\theta^T}$ is continuous on Θ.

(A5) For all $t, \sup_{\theta\in\Theta}\left|\tilde{\lambda}_t(\theta) - \lambda_t(\theta)\right| \leq V\rho^t$ and

$$\sup_{\theta\in\Theta}\left\|\frac{\partial\tilde{\lambda}_t(\theta)}{\partial\theta} - \frac{\partial\lambda_t(\theta)}{\partial\theta}\right\| \leq V\rho^t.$$

(A6) $v^T\frac{\partial\lambda_1(\theta)}{\partial\theta} = 0$ a.s. if and only if $v = 0$.

Under **(A1)**–**(A6)**, we can see that as $n \to \infty, \hat{\theta}_n \to \theta_0$ *a.s.* and

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N\left(0, J^{-1}(\theta_0) I(\theta_0) J^{-1}(\theta_0)\right),$$

where

$$\begin{aligned} I(\theta_0) &= E\left(\frac{\partial \ell_t(\theta_0)}{\partial \theta}\frac{\partial \ell_t(\theta_0)}{\partial \theta^T}\right) \\ &= E\left(\frac{Y_t}{\lambda_t(\theta_0)} - 1\right)\left(\frac{\partial \lambda_t(\theta_0)}{\partial \theta}\frac{\partial \lambda_t(\theta_0)}{\partial \theta^T}\right), \\ J(\theta_0) &= -E\left(\frac{\partial^2 \ell_t(\theta_0)}{\partial \theta \partial \theta^T}\right) \\ &= E\left(\frac{1}{\lambda_t(\theta_0)}\frac{\partial \lambda_t(\theta_0)}{\partial \theta}\frac{\partial \lambda_t(\theta_0)}{\partial \theta^T}\right) \end{aligned}$$

with $\ell_t(\theta) = -\lambda_t(\theta) + Y_t \log \lambda_t(\theta)$. See Ahmad and Francq (2016). For example, when $\lambda_t(\theta)$ is linear with $\theta = (\omega, \alpha, \beta)^T, \frac{\partial \lambda_t(\theta)}{\partial \theta} = (1, \lambda_{t-1}, Y_{t-1})^T$.

Here, we consider the problem of testing the following hypotheses:

$H_0 : \theta$ remains the same over $Y_1, \ldots Y_n$ vs. H_1 not H_0

For this task, we employ the score vector-based CUSUM test given by

$$\hat{T}_n^{score} = \max_{1 \le k \le n} \frac{1}{n}\left(\sum_{t=1}^{k} \frac{\partial \tilde{\ell}_t(\hat{\theta}_n)}{\partial \theta}\right)^T \hat{I}_n^{-1}\left(\sum_{t=1}^{k} \frac{\partial \tilde{\ell}_t(\hat{\theta}_n)}{\partial \theta}\right), \quad (2)$$

where

$$\hat{I}_n = \frac{1}{n}\sum_{t=1}^{n}\left(\frac{Y_t}{\tilde{\lambda}_t(\hat{\theta}_n)} - 1\right)^2\left(\frac{\partial \tilde{\ell}_t(\hat{\theta}_n)}{\partial \theta}\frac{\partial \tilde{\lambda}_t(\hat{\theta}_n)}{\partial \theta^T}\right).$$

The critical values of $\widehat{T}_n^{score}$ are obtained asymptotically from the following.

Theorem 1

Suppose that **(A1)**–**(A6)** are fulfilled. Then, under H_0, as $n \to \infty$,

$$\widehat{T}_n^{score} \xrightarrow{d} \sup_{0 \le S \le 1} \left\| B_d^{\circ}(S) \right\|^2,$$

where $\{B_d^{\circ}(S), 0 < s < 1\}$ is a d–dimensional Brownian bridge.

Proof. We first verify

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} \left\| \tilde{S}_k(\hat{\theta}_n) - \left\{ S_k(\theta_0) - \frac{k}{n} S_n(\theta_0) \right\} \right\| = o_p(1), \quad (3)$$

where

$$\tilde{S}_k(\theta) = \sum_{t=1}^{k} \frac{\partial \tilde{\ell}_t(\theta)}{\partial \theta} \text{ and } S_k(\theta) = \sum_{t=1}^{k} \frac{\partial \ell_t(\theta)}{\partial \theta}.$$

By **(A5)**, we have

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} | \tilde{S}_k(\hat{\theta}_n) - S_k(\hat{\theta}_n) | = o_p(1).$$

Thus, it suffices to show that

$$\frac{1}{\sqrt{n}} \max_{1 \le k \le n} \| \tilde{S}_k(\hat{\theta}_n) - S_k(\hat{\theta}_n) - \frac{k}{n}\{ S_n(\hat{\theta}_n) - S_n(\theta_n)\} \| = o_p(1).$$

By the mean value theorem, we get

$$S_k(\hat{\theta}_n) = S_k(\theta_n) + \sum_{t=1}^{k} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T}(\hat{\theta}_n - \theta_0)$$

for some intermediate point θ_n^* between θ_0 and $\hat{\theta}_n$. Thus, for any sequence of positive integers ζ_n with $\zeta_n \to \infty$ and $\frac{\zeta_n}{n} \to 0$ as $n \to \infty$, we have

$$\begin{aligned} &\frac{1}{\sqrt{n}} \max_{1 \le k \le n} \left\| S_k(\hat{\theta}_n) - S_k(\theta_0) - \frac{k}{n}\{ S_n(\hat{\theta}_0) - S_n(\theta_0)\} \right\| \\ &\le \frac{1}{\sqrt{n}} \max_{1 \le k \le \zeta_n} \left\| \sum_{t=1}^{k} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T}(\hat{\theta}_n - \theta_0) - \frac{k}{n}\sum_{t=1}^{n} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T}(\tilde{\theta}_n - \theta_0) \right\| \\ &+ \frac{1}{\sqrt{n}} \max_{\zeta_n \le k \le n} \left\| \sum_{t=1}^{k} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T}(\hat{\theta}_n - \theta_0) - \frac{k}{n}\sum_{t=1}^{k} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T}(\hat{\theta}_n - \theta_0) \right\| \\ &\le \left\{ \frac{1}{n}\sum_{t=1}^{\zeta_n} \left\| \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T} \right\| + \frac{\zeta_n}{n}\frac{1}{n}\sum_{t=1}^{n} \left\| \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T} \right\| \right\} . O_p(1) \\ &+ \max_{\zeta_n \le k \le n} \left\{ \left\| \frac{1}{k}\sum_{t=1}^{k} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T} - \frac{1}{n}\sum_{t=1}^{n} \frac{\partial^2 \ell_t(\theta_n^*)}{\partial \theta \partial \theta^T} \right\| \right\} . O_p(1), \end{aligned}$$

which can be seen to be $o_p(1)$ by using the ergodicity, **(A4)** and the dominated convergence theorem. Thus, Equation 3 is asserted.

Next, because $\left\{ \frac{\partial \ell_t(\theta_0)}{\partial \theta}, \mathcal{F}_t \right\}$ is a sequence of stationary martingale differences, using Donsker's invariance principle, we have

$$I(\theta_0)^{-1/2}\frac{1}{\sqrt{n}}S_{[ns]}(\theta_0)\xrightarrow{d}B_d(s),$$

Hence, owing to Equation 3, we have

$$I(\theta_0)^{-1/2}\frac{1}{\sqrt{n}}\tilde{S}_{[ns]}(\theta_0)\xrightarrow{d}B_d(s),$$

Since $\hat{I}_n$ converges to $I_n(\theta_0)$, we obtain

$$\hat{I}_n^{-\frac{1}{2}}\frac{1}{\sqrt{n}}\tilde{S}_{[ns]}(\hat{\theta}_n)\xrightarrow{d}B_d^{\circ}(s),$$

This establishes the theorem.

We reject the null hypothesis at the level of 0.05 if $\hat{T}_n^{score}$ is above 3.004 in our simulation study. Instead of the score vector-based CUSUM test, we can also employ the residual-based CUSUM test of Kang and Lee (2014) and Lee and Lee (2019) as seen below when the conditional variance $V_t(\theta)$ parameterized with θ is also well formulated, as in the INGARCH and INAR models. In practice, however, it is impossible to derive the form of the conditional variance of INGARCH models without knowing their underlying distributions. As such, Lee and Lee's CUSUM test is applicable only when the underlying distribution is known.

Let

$$\epsilon_t = Y_t - \lambda_t, \hat{\epsilon}_t = Y_t - \tilde{\lambda}_t(\hat{\theta}_n),$$
$$\eta_t = (Y_t - \lambda_t)/\sqrt{V_t}, \hat{\eta}_t = \hat{\epsilon}_t/\sqrt{\tilde{V}_t(\hat{\theta}_n)}.$$

Kang and Lee (2014) and Lee and Lee (2019) considered the CUSUM tests $\hat{T}_n^{res}$ and $\hat{T}_n^{sres}$, based on the ordinary and standardized residuals as follows:

$$\hat{T}_n^{res} = \frac{1}{\sqrt{n}\hat{\tau}_{n1}}\max_{1\le k\le n}\left|\sum_{t=1}^{k}\hat{\varepsilon}_t - \frac{k}{n}\sum_{t=1}^{n}\hat{\varepsilon}_t\right|, \quad (4)$$

$$\hat{T}_n^{sres} = \frac{1}{\sqrt{n}\hat{\tau}_{n2}}\max_{1\le k\le n}\left|\sum_{t=1}^{k}\hat{\eta}_t - \frac{k}{n}\sum_{t=1}^{n}\hat{\eta}_t\right|, \quad (5)$$

where $\hat{\tau}_{n1}^2$ and $\hat{\tau}_{n2}^2$ are consistent estimators of $\tau_1^2 = Var(\varepsilon_t)$ and $\tau_2^2 = Var(\eta_t)$. Then, under the regularity conditions on λ_t and $\tilde{\lambda}_t$ as in Lee and Lee (2019), $\hat{T}_n^{res}$ and $\hat{T}_n^{sres}$ converge weakly to $\sup_{0\le s\le 1}|B^{\circ}(s)|$ as $n\to\infty$, where B° is a Brownian bridge. The proof is omitted because it is essentially the same as that of Theorem 1 of Kang and Lee (2014) and Lee and Lee (2019). In particular, Lee and Lee (2019) demonstrated the superiority of $\hat{T}_n^{sres}$ to $\hat{T}_n^{res}$ in terms of power. In our simulation study, however, we merely consider $\hat{T}_n^{res}$ because V_t is unknown in general. We reject H_0 if $\hat{T}_n^{res} > 1.358$ at the level of 0.05.

Meanwhile, Lee (2019) considered the CUSUM of squares test based on $\hat{\varepsilon}_t^2$:

$$\hat{T}_n^{score} = \frac{1}{\sqrt{n}}\max_{1\le k\le n}\left|\sum_{t=1}^{k}\hat{\varepsilon}_t^2 - \frac{k}{n}\sum_{t=1}^{n}\hat{\varepsilon}_t^2\right|.$$

Then, we can see that as $n\to\infty, \hat{T}_n^{score}/\sigma$ converges to the sup of a Brownian bridge in distribution, where

$$\sigma^2 = \gamma(0) + 2\sum_{h=1}^{\infty}\gamma(h) \in (0,\infty)$$

with $\gamma(h) = Cov(\varepsilon_1^2, \varepsilon_{1+h}^2)$. Moreover, we can see that as $n \to \infty$,

$$\sigma_n^2 := Var\left(n^{-1/2}\sum_{t-1}^{n}(\epsilon_t^2 - E\epsilon_t^2)\right) \to \sigma^2,$$

which can be proven similarly to Theorem 1 of Lee (2019) (see also the Appendix and Remark 1 (4) therein) using the β-mixing property of $\{Y_t\}$ and near epoch property of $\{\varepsilon_1^2\}$ (Davis and Liu (2016) and Woodridge and White (1988)). In practice, σ^2 must be estimated from the data. As an estimate of σ^2, we use

$$\hat{\sigma}_n^2 = \hat{\gamma}_n(0) + 2\sum_{h=1}^{h_n}\hat{\gamma}_n(h),$$
$$\hat{\gamma}_n(h) = \frac{1}{n}\sum_{t=1}^{n-h}\left(\hat{\varepsilon}_n^2 - \overline{\varepsilon^2}\right)\left(\hat{\varepsilon}_{n+h}^2 - \overline{\varepsilon^2}\right),$$
$$\overline{\varepsilon^2} = \frac{1}{n}\sum_{t=1}^{n}\hat{\varepsilon}_t^2$$

with $h_n = \sqrt{2}(\log_{10} n)^2$. See Lee (2019).

Our primary objective here is to evaluate the score vector-based CUSUM test $\hat{T}_n^{score}$ as well as the residual-based CUSUM tests $\hat{T}_n^{res}$ and $\hat{T}_n^{squares}$ calculated from the Poisson QMLE. In our simulation study, we consider the Poisson and NB-INGARCH(1,1) models and INAR(1) models for this task.

Example 1 (INGARCH(1,1) model): The Poisson INGARCH(1,1) model is given by:

$$Y_t \mid \mathcal{F}_{t-1} \sim Poisson(\lambda_t), \quad \lambda_t = \omega + \alpha\lambda_{t-1} + \beta Y_{t-1}$$

with $\omega>0$, α, $\beta\geq0$ and $\alpha+\beta<1$.

The zero-inflated Poisson INGARCH(1,1) model is given by

$$Y_t \mid \mathcal{F}_{t-1} \sim \rho\delta_0 + (1-\rho)\ Poisson\ (\lambda_t\ /\ (1-\rho)),$$

where δ_0 is the distribution with mass 1 at 0, ρ is a real number in (0,1), and λ_t is the same as above. Lee *et al*. (2016) verified that $\{Y_t\}$ is strictly stationary and ergodic. In addition, it can be checked that $\{Y_t\}$ is β-mixing. In particular, $Var(Y_t \mid \mathcal{F}_{t-1}) = (1-\rho)\lambda_t\,(1+\rho\lambda_t)$.

The NB-INGARCH(1,1) model is defined as

$$Y_t \mid \mathcal{F}_{t-1} \sim NB(r, p_t),$$
$$\lambda_t = \frac{r(1-p_t)}{p_t} = \omega + \alpha\lambda_{t-1} + \beta Y_{t-1},$$

where r is a positive integer and $NB(r, p)$ denotes the NB distribution with the probability mass function:

$$P(Y=k) = \binom{k+r-1}{r-1}(1-p)^k p^r, k = 0,1,2,...$$

In practice, r is unknown and must be estimated form data, for instance, using the Akaike Information Criterion (AIC) or Bayesian Information Criterion (BIC) as in Davis and Wu (2009).

The COM-INGARCH(1,1) model is defined by

$$Y_t \mid \mathcal{F}_{t-1} \sim COM(\lambda_t, v),$$
$$\lambda_t = \omega + \alpha\lambda_{t-1} + \beta Y_{t-1},$$

where COM (λ, v) denotes the COM distribution with the probability mass function:

$$P(Y=k) = \left(\frac{\mu^k}{k!}\right)^r \frac{1}{S(\mu,v)}$$

with $S(\mu,v) = \left(\sum_{k=0}^{\infty} \mu^k / k!\right)^v$.

Example 2 (INAR(1) model): The INAR(1) model is defined as:

$$Y_t = \phi \circ Y_{t-1} + Z_t,$$

where $\circ$ is a binomial thinning operator and Z_t are iid random variables following the Poisson or NB distribution. In this case, $\lambda_t(\theta) = \phi Y_{t-1} + \omega$ where $\omega = EZ_t$. As discussed by Ahmad and Francq (2016), the Poisson QMLE method provides an estimating tool that adopts a likelihood approach that can replace the conditional least squares method.

3. Simulation Study

In this section, we evaluate the three tests, namely, $\hat{T}_n^{score}$, $\hat{T}_n^{res}$ and $\hat{T}_n^{squares}$ in Equations 2, 4, 6 for the INGARCH(1,1) and INAR(1) models as follows:

(1) Poisson INGARCH(1,1) model
(2) NB-INGARCH(1,1) model
(3) Zero-inflated Poisson INGARCH(1,1) model
(4) COM Poisson INGARCH(1,1)
(5) Poisson INAR(1) model

In this simulation, the empirical sizes and powers are obtained as the rejection number of H_0 out of 1000 replications at the level of 0.05 for n = 500, 1000 and various parameter settings. The corresponding critical value is 1.358. To check the power, we consider the situation in which a parameter change occurs at $[n/2]$. Here, we use *R* running on Windows 10 and the package compoisson.

The results in Tables 1-5, wherein the bold face stands for the model with the largest power, show that there is no severe size distortion in most cases and that the size approaches 0.05 as the sample size increases. The power also increases as the sample size rises. Overall, the results reveal that $\hat{T}_n^{score}$ performs reasonably well in Cases 1–4 but over sizes in Case 5. In Case 1, $\hat{T}_n^{squares}$ mostly outperforms the others as seen in Table 1, whereas $\hat{T}_n^{score}$ also compares well with $\hat{T}_n^{squares}$, especially when n = 1000. In Case 2, as seen in Table 2, $\hat{T}_n^{score}$ appears to outperform the others to a large extent, which shows the efficacy of the score vector-based CUSUM test when the underlying distribution is not Poisson. In Case 3, only $\hat{T}_n^{squares}$ appears to be able to detect a change of zero proportion as seen in Table 3. Except for this case, $\hat{T}_n^{score}$ mostly outperforms the others as in Case 2. As seen in Table 4, the result in Case 4 also confirms the superiority of $\hat{T}_n^{score}$ to the others in terms of power, except for the v change case, in which $\hat{T}_n^{squares}$ performs better. However, $\hat{T}_n^{score}$ appears to oversize in some cases. In Case 5, as seen in Table 5, $\hat{T}_n^{score}$ oversizes and $\hat{T}_n^{res}$ appears to outperform the others in most cases. Moreover, it turns out that $\hat{T}_n^{squares}$ detects a change in the innovation variance ω well, whereas it cannot detect a change in the thinning parameter ϕ. Our findings show that none of the tests completely outperforms the other tests and that $\hat{T}_n^{score}$ is highly recommended for INGARCH(1,1) models. However, when the zero proportional change is our main interest, $\hat{T}_n^{squares}$ is preferred. Moreover, when dealing with INAR models, we recommend using $\hat{T}_n^{res}$.

Table 1: Sizes and powers for Poisson INGARCH(1,1) model

(ω, α, β)		size			power		
$\to (\omega', \alpha', \beta')$	n	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$
(1, 0.2, 0.3)	500	0.048	0.027	0.031	**0.992**	0.903	0.922
→ (2, 0.2, 0.3)	1000	0.042	0.040	0.043	**1.000**	0.977	0.999
(1, 0.2, 0.3)	500	0.048	0.027	0.031	**0.983**	0.967	0.822
→ (0.5, 0.2, 0.3)	1000	0.042	0.040	0.043	**1.000**	0.996	0.998
(0.5, 0.2, 0.3)	500	0.043	0.026	0.034	**0.990**	0.963	0.849
→ (1, 0.2, 0.3)	1000	0.041	0.036	0.035	**1.000**	0.995	0.998
(1, 0.2, 0.3)	500	0.048	0.027	0.031	0.555	0.335	**0.999**
→ (1, 0.6, 0.3)	1000	0.042	0.040	0.043	0.933	0.908	**1.000**
(1, 0.2, 0.3)	500	0.048	0.027	0.031	0.817	**0.991**	0.966
→ (1, 0.2, 0.7)	1000	0.042	0.040	0.043	**1.000**	**1.000**	0.999
(1, 0.2, 0.3)	500	0.048	0.027	0.031	0.560	0.972	**0.983**
→ (1, 0.3, 0.6)	1000	0.042	0.040	0.043	**1.000**	**1.000**	**1.000**
(1, 0.3, 0.6)	500	0.036	0.026	0.066	0.163	0.465	**0.770**
→ (2, 0.3, 0.6)	1000	0.050	0.029	0.053	0.855	0.971	**0.978**
(1, 0.3, 0.6)	500	0.036	0.026	0.066	0.124	0.358	**0.638**
→ (0.5, 0.3, 0.6)	1000	0.050	0.029	0.053	0.685	0.851	**0.904**
(0.5, 0.3, 0.6)	500	0.045	0.029	0.044	0.178	0.253	**0.583**
→ (1, 0.3, 0.6)	1000	0.038	0.027	0.059	0.787	0.762	**0.894**
(1, 0.3, 0.6)	500	0.036	0.026	0.066	0.696	**0.995**	0.936
→ (1, 0.1, 0.6)	1000	0.050	0.029	0.053	**1.000**	**1.000**	**1.000**
(1, 0.3, 0.6)	500	0.036	0.026	0.066	0.968	0.979	**0.994**
→ (1, 0.3, 0.1)	1000	0.050	0.029	0.053	**1.000**	**1.000**	**1.000**
(1, 0.3, 0.6)	500	0.036	0.026	0.066	0.789	0.990	**0.992**
→ (1, 0.2, 0.3)	1000	0.050	0.029	0.053	**1.000**	**1.000**	**1.000**

Table 2: Sizes and powers for negative binomial INGARCH(1,1) model

$(\omega, \alpha, \beta, r)$		size			power		
$\to (\omega', \alpha', \beta', r')$	n	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$
(1, 0.2, 0.3, 1)	500	0.049	0.030	0.023	**0.896**	0.763	0.481
→ (2, 0.2, 0.3, 1)	1000	0.040	0.032	0.017	**0.999**	0.988	0.819
(1, 0.2, 0.3, 1)	500	0.049	0.030	0.023	**0.768**	0.044	0.401
→ (1, 0.6, 0.3, 1)	1000	0.040	0.032	0.017	**0.994**	0.186	0.617
(1, 0.2, 0.3, 1)	500	0.049	0.030	0.023	**0.313**	0.082	0.102
→ (1, 0.2, 0.7, 1)	1000	0.040	0.032	0.017	**0.659**	0.260	0.182
(1, 0.2, 0.3, 1)	500	0.049	0.030	0.023	**0.466**	0.098	0.142
→ (1, 0.3, 0.6, 1)	1000	0.040	0.032	0.017	**0.889**	0.303	0.240
(1, 0.2, 0.3, 3)	500	0.044	0.033	0.025	**0.991**	0.937	0.831
→ (2, 0.2, 0.3, 3)	1000	0.035	0.043	0.022	**1.000**	0.992	0.984
(1, 0.2, 0.3, 3)	500	0.044	0.033	0.025	0.493	0.018	**0.809**
→ (1, 0.6, 0.3, 3)	1000	0.035	0.043	0.022	**0.961**	0.236	0.935
(1, 0.2, 0.3, 3)	500	0.044	0.033	0.025	**0.498**	0.328	0.234
→ (1, 0.2, 0.7, 3)	1000	0.035	0.043	0.022	**0.924**	0.641	0.408
(1, 0.2, 0.3, 3)	500	0.044	0.033	0.025	**0.550**	0.296	0.358
→ (1, 0.3, 0.6, 3)	1000	0.035	0.043	0.022	**0.962**	0.647	0.516
(1, 0.3, 0.6, 3)	500	0.036	0.013	0.013	**0.491**	0.009	0.079
→ (2, 0.3, 0.6, 3)	1000	0.035	0.020	0.018	**0.884**	0.040	0.172
(1, 0.3, 0.6, 3)	500	0.036	0.013	0.013	**0.514**	0.299	0.168
→ (1, 0.1, 0.6, 3)	1000	0.035	0.020	0.018	**0.953**	0.773	0.387
(1, 0.3, 0.6, 3)	500	0.036	0.013	0.013	**0.719**	0.662	0.307
→ (1, 0.3, 0.1, 3)	1000	0.035	0.020	0.018	**0.983**	0.912	0.535
(1, 0.3, 0.6, 3)	500	0.036	0.013	0.013	**0.634**	0.629	0.292
→ (1, 0.2, 0.3, 3)	1000	0.035	0.020	0.018	**0.980**	0.900	0.524

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Table 3: Sizes and powers for zero-inflated Poisson INGARCH(1,1) model

$(\omega, \alpha, \beta, \rho)$		size			power		
$\rightarrow (\omega', \alpha', \beta', \rho')$	n	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	0.992	0.953	**1.000**
→ (2, 0.2, 0.3, 0.2)	1000	0.040	0.029	0.035	**1.000**	0.992	**1.000**
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	0.619	0.037	**0.996**
→ (1, 0.6, 0.3, 0.2)	1000	0.040	0.029	0.035	0.986	0.394	**1.000**
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	**0.684**	0.301	0.366
→ (1, 0.2, 0.7, 0.2)	1000	0.040	0.029	0.035	**0.987**	0.755	0.575
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	**0.747**	0.340	0.545
→ (1, 0.3, 0.6, 0.2)	1000	0.040	0.029	0.035	**0.993**	0.826	0.767
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	0.034	0.037	**0.126**
→ (1, 0.2, 0.3, 0.3)	1000	0.040	0.029	0.035	0.036	0.031	**0.307**
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	0.048	0.051	**0.433**
→ (1, 0.2, 0.3, 0.4)	1000	0.040	0.029	0.035	0.056	0.041	**0.842**
(1, 0.2, 0.3, 0.2)	500	0.044	0.032	0.025	0.062	0.050	**0.677**
→ (1, 0.2, 0.3, 0.5)	1000	0.040	0.029	0.035	0.051	0.054	**0.979**
(1, 0.2, 0.3, 0.5)	500	0.048	0.037	0.030	0.061	0.046	**0.696**
→ (1, 0.2, 0.3, 0.2)	1000	0.049	0.035	0.027	0.068	0.067	**0.981**
(1, 0.3, 0.6, 0.2)	500	0.035	0.033	0.019	**0.669**	0.016	0.224
→ (2, 0.3, 0.6, 0.2)	1000	0.058	0.035	0.024	**0.968**	0.056	0.458
(1, 0.3, 0.6, 0.2)	500	0.035	0.033	0.019	**0.745**	0.424	0.440
→ (1, 0.1, 0.6, 0.2)	1000	0.058	0.035	0.024	**0.996**	0.952	0.698
(1, 0.3, 0.6, 0.2)	500	0.035	0.033	0.019	**0.893**	0.781	0.556
→ (1, 0.3, 0.1, 0.2)	1000	0.058	0.035	0.024	**1.000**	0.992	0.761
(1, 0.3, 0.6, 0.2)	500	0.035	0.033	0.019	**0.825**	0.753	0.541
→ (1, 0.2, 0.3, 0.2)	1000	0.058	0.035	0.024	**1.000**	0.991	0.757

Table 4: Sizes and powers for COM Poisson INGARCH(1,1) model

$(\omega, \alpha, \beta, v)$		size			power		
$\rightarrow (\omega', \alpha', \beta', v')$	n	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$
(1, 0.2, 0.3, 2)	500	0.086	0.018	0.040	**0.717**	0.265	0.401
→ (2, 0.2, 0.3, 2)	1000	0.071	0.031	0.039	**0.962**	0.201	0.744
(1, 0.2, 0.3, 2)	500	0.086	0.018	0.040	**0.648**	0.154	0.593
→ (1, 0.6, 0.3, 2)	1000	0.071	0.031	0.039	**0.935**	0.135	**0.935**
(1, 0.2, 0.3, 2)	500	0.086	0.018	0.040	0.399	**0.504**	0.118
→ (1, 0.2, 0.7, 2)	1000	0.071	0.031	0.039	0.835	**0.860**	0.242
(1, 0.2, 0.3, 2)	500	0.086	0.018	0.040	0.539	**0.657**	0.190
→ (1, 0.3, 0.6, 2)	1000	0.071	0.031	0.039	**0.957**	0.943	0.396
(1, 0.2, 0.3, 2)	500	0.086	0.018	0.040	0.971	0.739	**1.000**
→ (1, 0.2, 0.3, 1)	1000	0.071	0.031	0.039	**1.000**	0.818	**1.000**
(1, 0.2, 0.3, 1)	500	0.047	0.027	0.031	0.990	0.762	**0.996**
→ (1, 0.2, 0.3, 2)	1000	0.044	0.041	0.042	**1.000**	0.843	**1.000**
(1, 0.3, 0.6, 2)	500	0.055	0.025	0.032	**0.849**	0.717	0.300
→ (2, 0.3, 0.6, 2)	1000	0.062	0.036	0.036	**1.000**	0.854	0.622
(1, 0.3, 0.6, 2)	500	0.055	0.025	0.032	0.400	**0.491**	0.123
→ (1, 0.1, 0.6, 2)	1000	0.062	0.036	0.036	0.749	**0.848**	0.258
(1, 0.3, 0.6, 2)	500	0.055	0.025	0.032	**0.702**	0.592	0.179
→ (1, 0.3, 0.1, 2)	1000	0.062	0.036	0.036	**0.987**	0.827	0.425
(1, 0.3, 0.6, 2)	500	0.055	0.025	0.032	0.573	**0.652**	0.172
→ (1, 0.2, 0.3, 2)	1000	0.062	0.036	0.036	**0.952**	0.950	0.354
(1, 0.3, 0.6, 2)	500	0.055	0.025	0.032	0.480	0.496	**0.994**
→ (1, 0.3, 0.6, 1)	1000	0.062	0.036	0.036	0.965	0.994	**1.000**
(1, 0.3, 0.6, 1)	500	0.047	0.025	0.037	0.670	0.736	**0.998**
→ (1, 0.3, 0.6, 2)	1000	0.036	0.031	0.058	0.999	0.998	**1.000**

Table 5: Sizes and powers for Poisson INAR model

(ϕ, ω)		size			power		
$\to (\phi', \omega')$	n	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$	$\hat{T}_n^{score}$	$\hat{T}_n^{res}$	$\hat{T}_n^{squares}$
(0.3, 3)	500	0.131	0.028	0.038	0.989	**0.997**	0.069
→ (0.5, 3)	1000	0.138	0.037	0.036	**1.000**	**1.000**	0.122
(0.3, 3)	500	0.131	0.028	0.038	**1.000**	**1.000**	0.154
→ (0.7, 3)	1000	0.138	0.037	0.036	**1.000**	**1.000**	0.342
(0.3, 3)	500	0.131	0.028	0.038	**0.998**	0.726	0.781
→ (0.3, 5)	1000	0.138	0.037	0.036	**1.000**	0.768	0.994
(0.3, 3)	500	0.131	0.028	0.038	0.995	0.467	**1.000**
→ (0.3, 1)	1000	0.138	0.037	0.036	**1.000**	0.611	**1.000**
(0.7, 3)	500	0.130	0.029	0.047	**1.000**	**1.000**	0.060
→ (0.5, 3)	1000	0.145	0.024	0.038	**1.000**	**1.000**	0.081
(0.7, 3)	500	0.130	0.029	0.047	**1.000**	**0.999**	0.162
→ (0.3, 3)	1000	0.145	0.024	0.038	**1.000**	**1.000**	0.353
(0.7, 3)	500	0.130	0.029	0.047	0.998	**1.000**	0.812
→ (0.7, 5)	1000	0.145	0.024	0.038	**1.000**	**1.000**	0.996
(0.7, 3)	500	0.130	0.029	0.047	0.901	**1.000**	**1.000**
→ (0.7, 1)	1000	0.145	0.024	0.038	**1.000**	**1.000**	**1.000**
(0.3, 7)	500	0.116	0.022	0.039	**1.000**	0.998	0.084
→ (0.5, 7)	1000	0.116	0.038	0.041	**1.000**	**1.000**	0.116
(0.3, 7)	500	0.116	0.022	0.039	0.964	**0.996**	0.187
→ (0.7, 7)	1000	0.116	0.038	0.041	**1.000**	**1.000**	0.485
(0.3, 7)	500	0.116	0.022	0.039	**1.000**	0.718	0.513
→ (0.3, 10)	1000	0.116	0.038	0.041	**1.000**	0.761	0.880
(0.3, 7)	500	0.116	0.022	0.039	**0.999**	0.859	0.454
→ (0.3, 5)	1000	0.116	0.038	0.041	**1.000**	0.896	0.828

4. Conclusion

In this study, we considered the Poisson QMLE-based CUSUM tests using score vectors and residuals and compared their performance for INGARCH(1,1) and INAR(1) models through a simulation study. We deduced the limiting null distribution of the score vector-based CUSUM test under certain conditions. Our findings in the simulations showed that the QMLE-based CUSUM test using score vectors can serve as a promising alternative to the MLE-based CUSUM tests when the underlying distribution is unspecified. Moreover, the residual-based CUSUM of squares test and residual-based CUSUM test are suitable for the detection of the zero proportional change and parameter change in INAR models, respectively.

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7. Ethics

This article is original and contains unpublished resources. All the authors have read and sanctioned the manuscript and are gratified that there are no ethical concerns involved.

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