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Measure of Departure from Conditional Partial Symmetry for Square Contingency Tables

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Abstract: For square contingency tables, a symmetry of conditional probability in part of the off-diagonal cells is considered with respect to the main diagonal. A measure is proposed to represent the degree of departure from the symmetric structure and gives the approximate confidence interval. Some numerical experiments are performed to illustrate the validity and usefulness of the proposed measure. Japanese occupational data are analyzed using the measure.

Keywords: Conditional Symmetry, Square Contingency Table, Measure

Introduction

For square contingency tables, various symmetry and asymmetry with respect to the main diagonal have been studied (Agresti, 2013, Chap.11; Tahata and Tomizawa, 2014). Consider an $r \times r$ contingency table with the same row and column classifications. Let X and Y denote the row and column variables of the table, respectively. The Conditional Symmetry (CS) model is defined by:

$$Pr\left(X = i, Y = j \mid X < Y\right) = Pr\left(X = j, Y = i \mid X > Y\right) \quad (1)$$

for all (i, j) with $i < j$ (McCullagh, 1978). The CS model indicates a structure of symmetry of the conditional probability in the entire table. Let p_{ij} be the probability that an observation falls in the i^{th} row and j^{th} column of the table ($i = 1,..., r$; $j = 1,..., r$). The CS model can be expressed by $p_{ij} = \delta p_{ji}$ $(i \neq j)$. Thus, the CS model describes a proportional structure of cell probability with respect to the main diagonal with a constant proportion parameter δ. When $\delta = 1$, the CS model reduces to the Symmetry (S) model (Bowker, 1948). In data analysis, these models can give a simple interpretation when the models fit well. However, these models often fit the data poorly because the restrictions are strict. The Partial Symmetry (PS) model has been considered to relax the restriction of the S model (Saigusa *et al*., 2016). It indicates a symmetry of probability for at least one symmetrical cell pair, namely, $p_{ij} = \psi_{ij}$ $(i \neq j)$ where $\psi_{ab} = \psi_{ba}$ for at least one (a, b) with $a < b$.

A measure of the departure from the CS model defined above has been proposed (Tomizawa and Saitoh, 1999). The measure is based on the diversity index (Patil and Taillie, 1982) which has a special case of the Shannon entropy. Let $\Phi^{(\lambda)}$ denote the measure with the parameter λ for the CS model (see Appendix for the details). The measure for the PS model has been given by Saigusa *et al.* (2016). In data analysis, such a measure enables us to compare the departures from the corresponding model among different tables.

In this study, we consider a symmetric structure of the conditional probability in part of the off-diagonal cells instead of in the entire table. We are also interested in measuring how far the conditional probabilities are distant from those with the model.

Model and Measure

We consider a new model to relax the constraint of the CS model, defined by:

$$Pr\left(X = i, Y = j \mid X < Y\right) = Pr\left(X = j, Y = i \mid X > Y\right) \quad (2)$$

for at least one (a, b) with $a < b$. We call this the Conditional Partial Symmetry (CPS) model. This model describes that the conditional probability that an observation falls in cell (i, j) on the condition that $X < Y$ is equal to the conditional probability that the observation falls in cell (j, i) on the condition that $X > Y$ for at least one (i, j) with $i < j$. The CPS model explicitly includes the CS model as a special case.

We also consider a measure to represent the degree of departure from the CPS model. Let $\delta_U = \sum\sum_{s<t} p_{st}, \delta_L = \sum\sum_{s>t} p_{st}, p_{ij}^U = p_{ij}/\delta_U, p_{ji}^L = p_{ji}/\delta_L, q_{ij} = p_{ij}^U/\left(p_{ij}^U + p_{ji}^L\right)$ and $q_{ji} = p_{ji}^L/\left(p_{ij}^U + p_{ji}^L\right)$ for $i < j$. Assume that $\delta_U > 0$, $\delta_L > 0$ and $\{p_{ij} + p_{ji} > 0\}$. Noting that the CPS model can be expressed as $q_{ij} = q_{ji}$ for at least one (i, j) with $i < j$, consider the measure defined by:

$$\tau^{(\lambda)}=\prod_{i=1}^{r-1}\prod_{j=i+1}^{r}\left[1-\frac{\lambda 2^{\lambda}}{2^{\lambda}-1}H_{ij}^{(\lambda)}\right]^{\left(p_{ij}^{U}+p_{ij}^{L}\right)/2}\ (\lambda>-1) \tag{3}$$

where, $H_{ij}^{(\lambda)}=(1/\lambda)\left[1-\left(q_{ij}\right)^{\lambda+1}-\left(q_{ji}\right)^{\lambda+1}\right]$. The parameter is a real value chosen by the user. The measure $\tau^{(\lambda)}$ is given as the form of the weighted geometric mean $H_{ij}^{(\lambda)}$. The measure $\tau^{(\lambda)}$ satisfies some properties: for any λ (> -1):

(i) $0 \le \tau^{(\lambda)} \le 1$
(ii) $\tau^{(\lambda)} = 0$ if and only if the probability table has a structure of the CPS model
(iii) $\tau^{(\lambda)} = 1$ if and only if the degree of departure from the CPS model is the largest in meaning that $p^{c}_{ij} = 1$ (then $p^{c}_{ji} = 0$) or $p^{c}_{ji} = 1$ (then $p^{c}_{ij} = 0$) for all (i, j) with $i < j$, where $p^{c}_{ij} = p_{ij}/(p_{ij} + p_{ji})$ for $i \ne j$

Approximate Confidence Interval

Assuming that a multinomial distribution applies to the $r \times r$ table, an approximate confidence interval for $\tau^{(\lambda)}$ can be considered. Let n_{ij} be the observed frequency for (i, j) cell ($i = 1 ,..., r$; $j = 1 ,..., r$) and $n = \sum\sum n_{ij}$. The estimator of $\tau^{(\lambda)}$, denoted by $\hat{\tau}^{(\lambda)}$, is given by $\hat{\tau}^{(\lambda)}$ with replaced (p_{ij}) by $\left(\hat{p}_{ij}\right)$ where $\hat{p}_{ij} = n_{ij}/n$. Using the delta method (Agresti, 2013, p.587), $\sqrt{n}$ ($\hat{\tau}^{(\lambda)} - \tau^{(\lambda)}$) asymptotically has ($n \to \infty$) a normal distribution with mean 0 and variance $\sigma^2[\tau^{(\lambda)}]$, where:

$$\sigma^{2}\left[\tau^{(\lambda)}\right]=\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\ne i}}^{r}p_{ij}\left(\Omega_{ij}^{(\lambda)}\right)^{2}-\left[\sum_{i=1}^{r}\sum_{\substack{j=1\\ j\ne i}}^{r}p_{ij}\Omega_{ij}^{(\lambda)}\right]^{2}\ (\lambda>-1), \tag{4}$$

with

$$\Omega_{ij}^{(\lambda)}=\begin{cases}\dfrac{\tau^{(\lambda)}}{2\delta_{U}}\left[\log\xi_{ij}^{(\lambda)}-\dfrac{\omega_{ij}^{(\lambda)}}{\xi_{ij}^{(\lambda)}}\right.\\ \qquad\left.-\displaystyle\sum_{k=1}^{r-1}\sum_{l=k+1}^{r}p_{kl}^{U}\left(\log\xi_{kl}^{(\lambda)}-\dfrac{\omega_{kl}^{(\lambda)}}{\xi_{kl}^{(\lambda)}}\right)\right]\\ \qquad\qquad (i<j),\\ \dfrac{\tau^{(\lambda)}}{2\delta_{L}}\left[\log\xi_{ij}^{(\lambda)}-\dfrac{\omega_{ij}^{(\lambda)}}{\xi_{ij}^{(\lambda)}}\right.\\ \qquad\left.-\displaystyle\sum_{k=1}^{r-1}\sum_{l=k+1}^{r}p_{lk}^{L}\left(\log\xi_{kl}^{(\lambda)}-\dfrac{\omega_{lk}^{(\lambda)}}{\xi_{kl}^{(\lambda)}}\right)\right]\\ \qquad\qquad (i>j),\end{cases} \tag{5}$$

$$\xi_{ij}^{(\lambda)}=\begin{cases}1-\dfrac{2^{\lambda}}{2^{\lambda}-1}\left\{1-\left(q_{ij}\right)^{\lambda+1}-\left(q_{ji}\right)^{\lambda+1}\right\}(\lambda\ne 0),\\ 1-\dfrac{1}{\log 2}\left(-q_{ij}\log q_{ij}-q_{ji}\log q_{ji}\right)(\lambda\ne 0),\end{cases} \tag{6}$$

$$\omega_{ij}^{(\lambda)}=\begin{cases}\dfrac{2^{\lambda}(\lambda+1)}{2^{\lambda}-1}\left\{\left(q_{ji}\right)^{\lambda+1}-\left(q_{ij}\right)^{\lambda}q_{ji}\right\}(\lambda\ne 0),\\ \dfrac{1}{\log 2}\left(q_{ji}\log q_{ji}-q_{ji}\log q_{ij}\right)(\lambda=0).\end{cases} \tag{7}$$

Note that the asymptotic normal distribution $\sqrt{n}\left(\hat{\tau}^{(\lambda)}-\tau^{(\lambda)}\right)$ is applicable only when $0 < \tau^{(\lambda)} < 1$. Let $\hat{\sigma}^{2}\left[\tau^{(\lambda)}\right]$ be $\sigma^2[\tau^{(\lambda)}]$ with (p_{ij}) replaced by $\left(\hat{p}_{ij}\right)$. The approximate $100(1 - \alpha)$ % confidence interval of $\tau^{(\lambda)}$ is $\hat{\tau}^{(\lambda)} \pm z_{\alpha/2}\hat{\sigma}\left[\tau^{(\lambda)}\right]/\sqrt{n}$, where $z_{\alpha/2}$ is the upper $(\alpha/2)$th quantile of the standard normal distribution.

The confidence interval given above is unbounded by (0,1) although the value of $\tau^{(\lambda)}$ must lie between 0 and 1. Therefore, we also give a modified confidence interval of the measure $\tau^{(\lambda)}$ using log(−log) transformation (Lachin, 2009, p.17). Let $\theta^{(\lambda)} = \log[-\log(\tau^{(\lambda)})]$ and let $\hat{\theta}^{(\lambda)}$ denote the sample version of $\theta^{(\lambda)}$. When using the delta method, $\sqrt{n}\left(\hat{\theta}^{(\lambda)}-\theta^{(\lambda)}\right)$ asymptotically has ($n \to \infty$) a normal distribution which has a mean 0 and variance $\sigma^2[\theta^{(\lambda)}]$, where:

$$\sigma^{2}\left[\theta^{(\lambda)}\right]=\left(\frac{\sigma\left[\tau^{(\lambda)}\right]}{\tau^{(\lambda)}\log\tau^{(\lambda)}}\right)^{2}\ (\lambda>-1). \tag{8}$$

Hence, the modified version of the approximate confidence interval using the log (−log) transformation is then given by $\exp\left(-\exp\left(\hat{\theta}^{(\lambda)}\mp z_{\alpha/2}\hat{\sigma}\left[\theta^{(\lambda)}\right]/\sqrt{n}\right)\right)$, where $\hat{\sigma}\left[\theta^{(\lambda)}\right]$ is the sample version of $\sigma[\theta^{(\lambda)}]$. The confidence limits are bounded by (0,1).

Artificial Examples

We show the validity of the measure $\tau^{(\lambda)}$ for representing the departure from the CPS model. Table 1 gives the 4 × 4 artificial cell probability tables. The CS and CPS models hold in Table 1a. Table 1b has the structure of the CPS model with $q_{23} = q_{32}$ and $q_{st} \ne q_{ts}$ for $s < t$ with $(s, t) \ne (2,3)$. Namely, the symmetry of the conditional probability holds partially in (2,3) and (3,2) cells. In the order of Tables 1b to 1e, the probability p_{32} decreases, and the other off-diagonal probabilities remain. Therefore, the departure from the CPS model may be larger in the order of Tables 1b to 1e. Table 1f is considered for the subsequent simulation. Table 1g has a distribution that indicates the largest departure. The values of $\tau^{(\lambda)}$ and $\Phi^{(\lambda)}$ applied to each of Tables 1a-g are displayed in Tables 2 and 3. We observed that for any fixed λ, (i) the values of $\tau^{(\lambda)}$ for Tables 1a-b were equal to

0, (ii) the value of $\tau^{(\lambda)}$ increased in the order of Tables 1b to 1e, and (iii) the value of $\tau^{(\lambda)}$ for Table 1g was equal to 1. Therefore $\tau^{(\lambda)}$ might be reasonable to express the degree of departure from the CPS model. On the other hand, the values of measure $\Phi^{(\lambda)}$ for Tables 1a-b which the CPS model holds were unequal. Thus, the measure $\Phi^{(\lambda)}$ was not appropriate for the CPS model.

We also conducted a simulation to evaluate the coverage probability of the confidence intervals of $\tau^{(\lambda)}$. The simulated data were generated from each Table 1b, 1d, and 1e with sample sizes n = 500, 1,000, and 10,000. We adopted a confidence level of $1 - \alpha = 0.95$. The true values of measure $\tau^{(\lambda)}$ for the generating distributions are given in Table 2.

For example, we observed that $\tau^{(0)} = 0.136$ for Table 1b, $\tau^{(0)} = 0.511$ for Table 1d and $\tau^{(0)} = 0.848$ for Table 1e.

Table 4 shows the coverage probabilities from 100,000 times simulations. Especially, for the modified confidence interval, a sample size of about 1,000 may be required to get an adequate confidence interval.

Table 1: Cell probability Table

(a)

0.032	0.040	0.008	0.024
0.010	0.028	0.200	0.040
0.002	0.050	0.034	0.360
0.006	0.010	0.090	0.066

(b)

0.122	0.090	0.008	0.163
0.001	0.028	0.200	0.002
0.043	0.050	0.034	0.137
0.006	0.046	0.004	0.066

(c)

0.122	0.090	0.008	0.163
0.001	0.028	0.200	0.002
0.043	0.040	0.034	0.137
0.006	0.046	0.004	0.076

(d)

0.122	0.090	0.008	0.163
0.001	0.028	0.200	0.002
0.043	0.030	0.034	0.137
0.006	0.046	0.004	0.086

(e)

0.122	0.090	0.008	0.163
0.001	0.028	0.200	0.002
0.043	0.010	0.034	0.137
0.006	0.046	0.004	0.106

(f)

0.122	0.090	0.001	0.163
0.001	0.028	0.200	0.001
0.043	0.001	0.034	0.137
0.001	0.046	0.001	0.131

(g)

0.122	0.090	0.000	0.163
0.000	0.028	0.200	0.000
0.043	0.000	0.034	0.137
0.000	0.046	0.000	0.137

Table 2: Values of $\tau^{(\lambda)}$ for Tables 1a-g

	λ				
Applied tables	-0.5	0	0.5	1	1.5
Table 1a	0.000	0.000	0.000	0.000	0.000
Table 1b	0.000	0.000	0.000	0.000	0.000
Table 1c	0.093	0.136	0.156	0.165	0.167
Table 1d	0.173	0.252	0.288	0.304	0.307
Table 1e	0.363	0.511	0.575	0.601	0.607
Table 1f	0.689	0.848	0.894	0.908	0.910
Table 1g	1.000	1.000	1.000	1.000	1.000

Table 3: Values of $\Phi^{(\lambda)}$ for Tables 1a-g

	λ				
Applied tables	-0.5	0	0.5	1	1.5
Table 1a	0.000	0.000	0.000	0.000	0.000
Table 1b	0.334	0.446	0.489	0.504	0.507
Table 1c	0.347	0.462	0.506	0.522	0.525
Table 1d	0.365	0.485	0.530	0.547	0.551
Table 1e	0.435	0.577	0.633	0.654	0.659
Table 1f	0.707	0.856	0.898	0.911	0.913
Table 1g	1.000	1.000	1.000	1.000	1.000

Table 4: The values of coverage probability (CP and CP (log − log)) of the approximate confidence intervals based on the variances in (4) and (8), obtained from generating 100,000 simulation data sets with sample sizes n for Tables 1b, 1d and 1e

(a) For Table 1b

n	λ	CP	CP (log − log)
500	0.0	0.959	0.944
	0.5	0.957	0.946
	1.0	0.956	0.947
1,000	0.0	0.964	0.948
	0.5	0.962	0.949
	1.0	0.961	0.950
5,000	0.0	0.968	0.958
	0.5	0.967	0.957
	1.0	0.967	0.958

(b) For Table 1d

n	λ	CP	CP (log − log)
500	0.0	0.940	0.958
	0.5	0.942	0.960
	1.0	0.943	0.960
1,000	0.0	0.945	0.953
	0.5	0.946	0.955
	1.0	0.946	0.954
5,000	0.0	0.950	0.951
	0.5	0.950	0.951
	1.0	0.950	0.951

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(c) For Table 1e

n	λ	CP	CP (log − log)
500	0.0	0.857	0.842
	0.5	0.862	0.878
	1.0	0.908	0.876
1,000	0.0	0.883	0.954
	0.5	0.902	0.953
	1.0	0.908	0.951
5,000	0.0	0.939	0.951
	0.5	0.942	0.952
	1.0	0.944	0.952

Table 5: Occupational status for Japanese father-son pairs (Tominaga, 1979, p.131)

(a)

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1)	80	72	37	19	208
(2)	44	155	61	31	292
(3)	26	73	218	45	362
(4)	69	156	166	614	1005
Total	219	456	482	709	1886

(b)

	Son's status				
Father's status	(1)	(2)	(3)	(4)	Total
(1)	127	101	54	12	294
(2)	86	207	125	13	431
(3)	78	124	310	24	536
(4)	109	206	437	325	1077
Total	400	638	926	374	2338

Note: Status (1) indicates Professional and Managers; (2) Clerical and Sales; (3) Skilled manual and Semiskilled manual; and (4) Unskilled manual and Farmers

Table 6: The estimates of $\tau(\lambda)$ ($\hat{\tau}^{(\lambda)}$), the Standard Error (SE) of $\hat{\tau}^{(\lambda)}$, the 95% confidence intervals (CI and CI (log − log)) based on the variances in (4) and (8), applied to each of Tables 5a and 5b

(a) For Table 5a

λ	$\hat{\tau}^{(\lambda)}$	SE	CI	CI (log − log)
−0.5	0.058	0.013	(0.033,0.083)	(0.036,0.087)
0	0.096	0.021	(0.055,0.136)	(0.060,0.141)
0.5	0.118	0.025	(0.068,0.168)	(0.074,0.173)
1	0.129	0.028	(0.075,0.183)	(0.081,0.188)
1.5	0.132	0.028	(0.077,0.187)	(0.083,0.192)

(b) For Table 5b

λ	$\hat{\tau}^{(\lambda)}$	SE	CI	CI (log − log)
−0.5	0.154	0.015	(0.124,0.184)	(0.125,0.185)
0	0.245	0.023	(0.200,0.289)	(0.201,0.290)
0.5	0.294	0.026	(0.243,0.346)	(0.244,0.347)
1	0.318	0.028	(0.263,0.371)	(0.264,0.372)
1.5	0.323	0.028	(0.268,0.378)	(0.269,0.378)

Real Data Examples

The data in Tables 5a and 5b are taken from Tominaga (1979, p.131). These tables show the cross-classification of a father's and his son's occupational status categories in Japan, which were examined in 1955 and 1975. The row and column categories are the father's and son's occupational status, respectively. The smaller category number corresponds to a higher status. Thus, the father-son pairs in the upper triangle mean the inter-generation social mobility to a lower status and the pairs in the lower triangle mean the social mobility to a higher status. As for Tables 5a and 5b, the CPS model indicates the conditional probability that the father's and son's statuses are i and j when there is social mobility to a lower status is equal to the conditional probability that the father's and son's statuses are j and i when there is social mobility to a higher status for at least one (i, j) with $i < j$. We apply the measure $\tau^{(\lambda)}$ to compare the departure from the CPS for Japanese social mobilities in 1955 and 1975. Tables 6a and 6b display the estimated values of the measure $\tau^{(\lambda)}$ applied to each of Tables 5a and 5b, respectively, and show the 95% confidence intervals. Comparing the confidence intervals for Tables 5a and 5b for any fixed λ (> -1), the degree of departure from the CPS model for Japanese father's and son's status may be larger in 1975 than in 1955.

Conclusion

The present paper has focused on the symmetry of the conditional probability in partial cells of the contingency table and has proposed a measure to represent the departure from the partial symmetry, whereas the previous studies have discussed investigating the symmetry of conditional probability in the entire contingency table. The proposed measure expresses the degree of departure from the CPS model which indicates symmetry of conditional probability for some symmetric cells. The measure enables us to compare the degrees of departure between different tables when square contingency table data are obtained in various scientific investigations as shown in the artificial and real data examples in Sections 4 and 5.

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Author's Contributions

Yusuke Saigusa: Methodology, project administration, formal analysis, software, writing-original draft preparation, review and editing.

Nobuki Fukumoto: Formal analysis, software, writing-original draft preparation, review and editing.

Tomoyuki Nakagawa: Formal analysis, writing-review and editing.

Sadao Tomizawa: Conceptualization, methodology, writing-review and editing.

Ethics

This article is original work. The corresponding author confirms that the other authors have approved the manuscript and no ethical issues are involved.

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Appendix

Let $\delta_U = \sum\sum_{s<t} p_{st}, \delta_L = \sum\sum_{s>t} p_{st}, p_{ij}^U = p_{ij} / \delta_U, p_{ji}^L = p_{ji} / \delta_L, q_{ij} = p_{ij}^U / \left(p_{ij}^U + p_{ji}^L\right)$ and $q_{ji} = p_{ji}^L / \left(p_{ij}^U + p_{ji}^U\right)$ for $i < j$. Assume that $\delta_U > 0$, $\delta_L > 0$ and $\{p_{ij} + p_{ji} > 0\}$. The measure for the CS model was given as follows (Tomizawa and Saitoh, 1999):

$$\Phi^{(\lambda)} = \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \frac{1}{2}\left(p_{ij}^U + p_{ji}^L\right)\left[1 - \frac{\lambda 2^\lambda}{2^\lambda - 1} H_{ij}^{(\lambda)}\right] \quad (\lambda > -1), \tag{9}$$

where, $H_{ij}^{(\lambda)} = (1/\lambda)\left[1 - \left(q_{ij}\right)^{\lambda+1} - \left(q_{ji}\right)^{\lambda+1}\right]$. The value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. The measure $\Phi^{(\lambda)}$ is formulated as the weighted arithmetic mean of the diversity index $H_{ij}^{(\lambda)}$.