Journal of Mathematics and Statistics

Original Research Paper

Elements of Formal Probabilistic Mechanics

[1]Farida Kachapova and [2]Ilias Kachapov

[1]*Department of Mathematical Sciences, Auckland University of Technology, New Zealand*
[2]*Examination Academic Services, University of Auckland, New Zealand*

Article history
Received: 07-08-2021
Revised: 25-01-2022
Accepted: 08-02-2022

Corresponding Author:
Farida Kachapova
Department of Mathematical Sciences, Auckland University of Technology, New Zealand
E-mail: farida.kachapova@aut.ac.nz

Abstract: In this study model of particle motion on a three-dimensional lattice is created using discrete random walk with small steps. A probability space of the particle trajectories is rigorously constructed. Unlike deterministic approach in classical mechanics, here probabilistic properties of particle movement are used to formally derive analogues of Newton's first and second laws of motion. Similar probabilistic models can potentially be applied to justify laws of thermodynamics in a consistent manner.

Keywords: Newton's Laws of Motion, Particle Trajectory on Lattice, Random Walk, Kolmogorov Extension Theorem

Introduction

This study is motivated by some unsolved problems in thermodynamics. Heat transfer is regarded as the transfer of kinetic energy from one particle to another when they collide. This process is time-reversible: if the direction of time (or the direction of velocity) is reversed, then kinetic energy is transferred from the second particle to the first one. Thus, heat transfer is considered a deterministic process (when initial conditions uniquely determine the future and past positions, velocity and other characteristics), therefore it follows laws of classical mechanics.

However, this is inconsistent with the fact that heat transfer at the macroscopic level is not time-reversible: heat can be transferred from a hot solid to a cold one but not back. Another example: When hot and cold liquids are mixed the result is a warm liquid, but the warm liquid cannot be separated into the hot and cold liquids. This inconsistency might be resolved if molecule movement is assumed to follow probabilistic rather than deterministic laws, because probabilistic processes are irreversible.

Molecule movement under probabilistic laws is studied in statistical mechanics (or more generally, statistical physics). There is extensive research in equilibrium mathematical statistical mechanics. As Lykov and Malyshev (2018) write: "The mathematical equilibrium statistical physics has only one axiom - Gibbs distribution, which defines all this science.... And this only axiom produced fantastic mathematical explosion of this science around 1950 – 2000."

Most phenomena in nature, including irreversible processes, are time-dependent and are the subject of the non-equilibrium statistical mechanics, where the axiomatic approach is also expected to be useful. Thus, one of the aims in the non-equilibrium statistical mechanics is to introduce few main axioms and based on these axioms, to construct a theory, which is consistent with Newton's laws of motion, and the rest of the classical mechanics at the macroscopic level.

Research in this direction is being conducted, including the study of relations between microscopic and global properties of physical objects in statistical mechanics. The monograph by Attard (2012) investigates analogies between dynamic and stationary systems and generalizes results from the equilibrium statistical mechanics to the time-dependent systems. Livi and Politi (2017) cover several modern approaches in non-equilibrium statistical physics, in particular non-equilibrium phase transitions. Lykov and Malyshev (2018) consider Hamiltonian dynamics of a system of N particles in n-dimensional real space and prove ergodicity of this system if at least one particle of the system collides with external particles at random times. The book of Ropke (2013) describes several approaches in the modern nonequilibrium statistical physics, including master equations, kinetic equations, thermodynamic Green's function and linear response theory.

The current paper contributes to this direction of research by using an axiomatic approach, so that Newton's mechanics can be derived from the probabilistic laws of particle movement. Here a model is created, where a particle moves randomly in small steps while its long-distance movement becomes approximately deterministic. A rigorous approach is used in construction of this model.

Science Publications

Probability Space

In this section a probability space (**Ω**, $\mathcal{F}$, *P*) is constructed. The following concepts from measure theory will be used; these well-known definitions are taken from Billingsley (2012).

Definition 2.1.

Suppose **Ω** is a non-empty set and $\mathcal{A}$ is a family of subsets of **Ω**. $\mathcal{A}$ is called an **algebra over Ω** if it has the properties:

(1) $\emptyset \in \mathcal{A}$,
(2) $B \in \mathcal{A} \Rightarrow B^c = \Omega \backslash B \in \mathcal{A}$,
(3) $B, C \in \mathcal{A} \Rightarrow B \cup C \in \mathcal{A}$.

Definition 2.2.

Suppose **Ω** *is a non-empty set and* $\mathcal{F}$ *is a family of subsets of* **Ω**. $\mathcal{F}$ *is called a* ***σ-algebra on*** **Ω** *if it has the properties:*

(1) $\emptyset \in \mathcal{F}$,
(2) $B \in \mathcal{F} \Rightarrow B^c = \Omega \setminus B \in \mathcal{F}$,
(3) *for any elements* $B_1, B_2, B_3, \ldots$ of $\mathcal{F}$, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

2.1. Sample Space and Sigma-Algebra of Events

Definitions 2.3, 2.5, 2.6, 2.10, 2.13 below are created by the authors as a part of the construction of the probability space (**Ω**, $\mathcal{F}$, *P*).

Definition 2.3.

1) Denote $\mathbb{N}_7 = \{0, 1, 2, 3, 4, 5, 6\}$. The **sample space** is chosen as the set $\mathbf{\Omega} = (\mathbb{N}_7)^{\mathbb{N}}$.

Thus, any element ω of **Ω** is an infinite sequence $\omega = (\omega_0, \omega_1, \omega_2, \ldots)$ of natural numbers between 0 and 6. Here ω_k denotes the *k*-th element of the sequence ω (k = 0, 1, 2, ...). Such a sequence represents **trajectory of a particle** on 3-dimensional lattice (details are given in Section 3).

2) A **hyperplane** is defined as a set of the form:

$$\boldsymbol{C(n, i)} = \{\omega \in \Omega : \omega_n = i\},$$

where $n \in \mathbb{N}$ and $i \in \mathbb{N}_7$.

For each $n \in \mathbb{N}$ there are seven distinct hyperplanes: $C(n, 0)$, $C(n, 1)$, $C(n, 2)$,$C(n, 3)$, $C(n, 4)$, $C(n, 5)$, $C(n, 6)$.

Proposition 2.4.

1) If $i \neq j$, then $C(n, i) \cap C(n, j) = \emptyset$.
2) For any $n \in \mathbb{N}$:

$$\bigcup_{j=0}^{6} C(n, j) = \Omega.$$

3) Complement of any hyperplane $C(n, i)$ is a union of the other six hyperplanes:

$$\left(C(n,i)\right)^c = \bigcup_{\substack{j=0 \\ j\neq 1}}^{6} C(n, j)$$

Proof of the proposition is obvious.

Definition 2.5.

1) Denote $\boldsymbol{\mathcal{F}_0}$ the algebra generated by all hyperplanes $C(n, i)$, where $n \in \mathbb{N}$ and $i \in \mathbb{N}_7$. This means that $\mathcal{F}_0$ is the least algebra over **Ω** containing all hyperplanes.
2) Denote $\boldsymbol{\mathcal{F}}$ the σ-algebra generated by $\mathcal{F}_0$, i.e. the least σ-algebra on **Ω** containing $\mathcal{F}_0$. Elements of $\mathcal{F}$ are called **events**.

Thus, **Ω** and $\mathcal{F}$ have been introduced. To complete the definition of the probability space (**Ω**, $\mathcal{F}$, *P*), it remains to construct a probability measure *P*.

*2.2. Constructing Probability Measure on (***Ω***,* $\mathcal{F}$*)*

2.2.1. Planes

Definition 2.6.

A **plane** is defined as a subset of **Ω** that can be represented as the following intersection of hyperplanes:

$$D[i_0, i_1, \ldots, i_n] = C(0, i_0) \cap C(1, i_1) \cap \ldots \cap C(n, i_n),$$

where $n \in \mathbb{N}$ and each $i_k \in \mathbb{N}_7$.

In particular, $D[i] = C(0, i)$. For each $n \in \mathbb{N}$ there are 7^{n+1} planes $D[i_0, i_1, \ldots, i_n]$.

Proposition 2.7.

1) If $D[i_0, i_1, \ldots, i_n] \cap D[j_0, j_1, \ldots, j_n] \neq \emptyset$, then $i_0 = j_0$, $i_1 = j_1, \ldots, i_n = j_n$ and $D[i_0, i_1, \ldots, i_n] = D[j_0, j_1, \ldots, j_n]$.
2) If $m < n$ and $D[i_0, i_1, \ldots, i_n] \cap D[j_0, j_1, \ldots, j_m] \neq \emptyset$, then $i_0 = j_0$, $i_1 = j_1, \ldots, i_m = j_m$ and $D[i_0, i_1, \ldots, i_n] \subseteq D[j_0, j_1, \ldots, j_m]$.
3) If planes D_1 and D_2 intersect, then $D_1 \subseteq D_2$ or $D_2 \subseteq D_1$.
4) Intersection of two planes is $\emptyset$ or a plane.

Proof

1) Suppose $D[i_0, i_1, \ldots, i_n] \cap D[j_0, j_1, \ldots, j_n] \neq \emptyset$. Then

$$\left[\bigcap_{k=0}^{n} C(k, i_k)\right] \cap \left[\bigcap_{k=0}^{n} C(k, i_k)\right] \neq \phi$$

and there exists $\omega \in \Omega$ such that for any $k = 0, 1, 2,..., n$, $i_k = \omega_k = j_k$. So $D[i_0, i_1,...,i_n] = D[j_0, j_1,...,j_n]$.

2) Suppose $m < n$ and $D[i_0, i_1,...,i_n] \cap D[j_0, j_1,...,j_m] \neq \emptyset$. Then

$$\left[\bigcap_{k=0}^{n} C(k,i_k)\right] \cap \left[\bigcap_{k=0}^{m} C(k,j_k)\right] \neq \phi$$

and some ω' belongs to this intersection. Then

$$\text{for any } k = 0,1,2,...,m,\ i_k = \omega'_k = j_k. \tag{1}$$

Suppose $\omega \in D[i_0, i_1,..., i_n]$. Then by (1), for any $k = 0, 1, 2, ..., m$, $\omega_k = i_k = j_k$. So $\omega \in D[j_0, j_1,..., j_m]$.

This proves $D[i_0, i_1,..., i_n] \subseteq D[j_0, j_1,..., j_m]$.

3) This follows from parts 1) and 2).

4) This follows from part 3).

Proposition 2.8.

1) Any element A of $\mathcal{F}_0$ is $\emptyset$ or can be represented as a finite union of planes.
2) Any element A of $\mathcal{F}_0$ is $\emptyset$ or can be represented as a finite union of pairwise disjoint planes.

Proof

1) By the definition of algebra $\mathcal{F}_0$ its elements are constructed in the following steps.

- $\emptyset \in \mathcal{F}_0$.
- Any hyperplane $C(n,i) \in \mathcal{F}_0$ ($n \in \mathbb{N}$, $i \in \mathbb{N}_7$).
- If $B \in \mathcal{F}_0$, then $B^c \in \mathcal{F}_0$.
- If $B, H \in \mathcal{F}_0$, then $B \cup H \in \mathcal{F}_0$.

Consider an arbitrary $A \in \mathcal{F}_0$. The statement will be proven by induction on construction of A.

a) Case $A = \emptyset$ is obvious.
b) Case $A = C(n, i)$. By Proposition 2.4.2)

$$\Omega = \bigcup_{j=0}^{6} C(n, j).$$

$$A = \underbrace{\Omega \cap \Omega .... \cap \Omega}_{n\,times} \cap C(n,i) =$$

$$= \left[\bigcup_{j_0=0}^{6} C(0, j_0)\right] \cap ... \cap \left[\bigcup_{j_{n-1}=0}^{6} C(n-1, j_{n-1})\right] \cap (Cn,1)$$

$$= \bigcup_{j_0=0}^{6} ... \bigcup_{j_{n-1}=0}^{6} \left[C\left(0, j_0\right) \cap ... \cap C\left(n-1, j_{n-1}\right) \cap C\left(n, i\right)\right]$$

$$= \bigcup_{j_0=0}^{6} ... \bigcup_{j_{n-1}=0}^{6} D\left[j_0,..., j_{n-1}, i\right]$$

is a finite union of planes.

c) First consider a complement of a plane.

$$\left(D\left[i_0,...,i_n\right]\right)^c = \bigcup_{k=0}^{n}\left(C\left(k,i_k\right)\right)^c = \bigcup_{k=0}^{n} \bigcup_{\substack{j_k\ 0 \\ j_k \neq i_k}}^{6} C\left(k, j_k\right)$$

by Proposition 2.4.3). It was proven in part b) that any hyperplane is a finite union of planes. So

$$\left(D[i_0,...,i_n]\right)^c \textit{ is a finite union of planes.} \tag{2}$$

Now consider the case $A = B^c$, where $B \in \mathcal{F}_0$. If $B = \emptyset$, then by Proposition 2.4.2), $A = \Omega = \bigcup_{i=0}^{6} C(0,i) = \bigcup_{i=0}^{6} D[i]$ is a finite union of planes.

Suppose $B \neq \emptyset$. Then by the inductive assumption B is a finite union of planes:

$$A = \Omega = \bigcup_{i=0}^{6} C(0,i) = \bigcup_{i=0}^{6} D[i]$$

By (2) each D_k^c is a finite union of planes:

$$D_k^C = \bigcup_{l_k=0}^{rk} D'l_k$$

So:

$$A = \bigcap_{k=0}^{s} \bigcup_{l1=0}^{r_1} D'l_k = \bigcup_{l_s=0}^{r_1} ... \bigcup_{l_s=0}^{r_1} \left(D'l_1 \cap ... \cap D'l_s\right).$$

By Proposition 2.7.4), each $D'_{l_1} \cap ... \cap D'_{l_1}$ is $\emptyset$ or a plane. Then A is $\emptyset$ or a finite union of planes.

d) Case $A = B \cup H$, where $B, H \in \mathcal{F}_0$.

By the inductive assumption B is a finite union of planes and so is H. Clearly, this is also true for A.

2) Suppose $A \neq \emptyset$. Then by part 1), $A = \bigcup_{k=0}^{n} D_k$, where each D_k is a plane. The statement will be proven by induction on n.

a) Case $n = 1$ is obvious.

b) Assume the statement holds for n and $A = \bigcup_{k=0}^{n+1} D_k$.

Denote $B = \bigcup_{k=0}^{n} D_k$. By the inductive assumption B can be represented as a finite union of pairwise disjoint planes:

$$B = \bigcup_{l=0}^{n} D'_l. \tag{3}$$

So:

$$A = \bigcup_{l=0}^{m} D_l^{'} \cup D_{n+1} \tag{4}$$

Denote $D_{l_1}^{'}, ..., D'_{lr}$ all the planes from the union (3) that intersect with D_{n+1}. (If there are no such planes, then the statement is proven.)

If $D_{n+1} \subseteq D'_{li}$ for some i = 1, 2,..., r, then $A = \bigcup_{l=0}^{m} D_l'$ and the statement is proven.

Otherwise each $D'_{l_i} \subseteq D_{n+1}$ by Proposition 2.7.3) and each D'_{l_i} can be removed from the union (4):

$$A = \bigcup_{\substack{l=0,\\ l\neq l_1,\\ ...,\\ l\neq l_r}}^{m} D_l' \cup D_{n+1};$$

getting a finite union of pairwise disjoint planes.

2.2.2. Defining P_0 on $\mathscr{F}_0$

Notation 2.9.

For each $n \in \mathbb{N}$ fix seven non-negative numbers $p_n(i)$, i = 0, 1, ..., 6, such that $\sum_{i=0}^{6} p_n(i) = 1$. These numbers will be fixed for the rest of the article.

A simple example of such numbers is when all $pn(i) = \frac{1}{7}$.

Definition 2.10.

(1) $P_0(\emptyset) = 0$.
(1) $P_0(C(n, i)) = p_n(i)$ $(n \in \mathbb{N}, i \in \mathbb{N}_7)$.
(2) *For any plane* $D[i_0, i_1,...,i_n] = C(0, i_0) \cap C(1, i_1) \cap ... \cap C(n, i_n)$:

$$P_0\left(D\left[i_0, i_1, ..., i_n\right]\right) = \prod_{k=0}^{n} P_0\left(C(k, i_k\right)$$

Lemma 2.11.

If a plane D equals a union of pairwise disjoint planes:

$$D = \bigcup_{j=0}^{k} D_j,$$

then $P_0(D) = \sum_{j=0}^{k} P_0(D_j)$.

Proof

For any plane X its length is defined by the following:

if $X = D[j_0, j_1,..., j_m]$, then ***length***$(X) = m$.

Denote $D = D[i_0, i_1,..., i_n]$. So $length(D) = n$.
Suppose

$$D = \bigcup_{j=0}^{k} D_j. \tag{5}$$

Denote $M = \{D_j : j = 0,1,...,k\}$. Next the following will be proven for any $X \in M$:

$$length(X) \geq n \text{ and}$$

$$X = D[i_0, i_1,...,i_n, i_{n+1},...,i_m] \text{ for some } i_{n+1},..., i_m. \tag{6}$$

Proof of (6).

Any $X \in M$ has the form: $X = D[j_0, j_1,...,j_m]$. $length(X) = m$.

Assume $m < n$. By (5):

$$X \subseteq D,$$

so $X \cap D \neq \emptyset$.
By Proposition 2.7.2), $D[i_0, i_1,..., i_n] \subseteq D[j_0, j_1,..., j_m]$, that is $D \subseteq X$. So $X = D$ and $m = n$, which contradicts the assumption $m < n$. $m \geq n$ is proven.

Thus, X has the form: $X = D[j_0, j_1,...,j_n, j_{n+1},...,j_m]$. Since $X \subseteq D$, by Proposition 2.7.1), 2), $j_0 = i_0$, $j_1 = i_1$,..., $j_n = i_n$. This completes the proof of (6).

Denote $q = \max\{length(X) : X \in M\}$. By (6), $q \geq n$. The equality:

$$P_0(D) = \sum_{j=0}^{k} P_0(D_j) \tag{7}$$

will be proven by induction on q.

Base step: $q = n$. In this case any $X \in M$ has the form $X = D[i_0, i_1,..., i_n] = D$. Since elements of M are pairwise disjoint, D is the only element of M. That makes equality (7) obvious.

Inductive step. Assume (7) holds for q. Let $\max\{length(X) : X \in M\} = q + 1$. Denote:

$M' = \{X \in M : length(X) \leq q\}$ and

$$M'' = \{X \in M : length(X) = q + 1\}.$$

Then $M = M' \cup M''$, $M'' \neq \emptyset$. Denote $I = \{(i_{n+1},..., i_q) : D[i_0, i_1,...,i_n, i_{n+1},..., i_q, l] \in M'' \text{ for some } l\}$.

Fix $(i_{n+1},..., i_q) \in I$. Let us prove:

$$\text{for any } l \in \mathbb{N}_7,\quad D[i_0, i_1,...,i_n, i_{n+1},..., i_q, l] \in M''. \tag{8}$$

By the definition of I, for some $l_0 \in \mathbb{N}_7$, $X =$

$= D[i_0, i_1,...,i_n, i_{n+1},..., i_q, l_0] \in M''$. Without loss of generality one can assume $l_0 = 6$ and prove (8) for $l = 5$ (the general case is similar).

Take $\omega = (i_0, i_1,..., i_n, i_{n+1},..., i_q, 5, 0, 0,...)$. Then $\omega \in D$. By (5), $\omega \in Y$ for some $Y \in M = M' \cup M''$. Two cases.

Case 1: $Y \in M'$. Then $Y \neq X$ and $Y = D[i_0,..., i_m]$, where $n \le m \le q$.

For $\omega' = (i_0, i_1, ..., i_n, i_{n+1}, ..., i_q, 6, 0, 0,...)$ it holds: $\omega' \in X \cap Y$. This contradicts the fact that elements of M are pairwise disjoint. Case 1 is impossible.

Case 2: $Y \in M''$. Since $\omega \in Y$, it holds:
$Y = D[i_0, i_1,...,i_n, i_{n+1},..., i_q, 5]$, that is
$D[i_0, i_1,...,i_n, i_{n+1},..., i_q, 5] \in M''$. This completes the proof of (8).

(8) implies the following: $M'' = \{D[i_0, i_1,...,i_n, i_{n+1},..., i_q, l] : (i_n, i_{n+1},..., i_q) \in I \text{ and } l \in \mathbb{N}_7\}$.

Since $P_0\left(D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q,l\right]\right) = P_0\left(D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q,l\right]\right).P_0\left(C(q+1,l)\right)$, it holds:

$$\sum_{X\in M''} P_0(X) = \sum_{(i_n,i_{n+1},...,i_q)\in I}\sum_{l=0}^{6} P_0\left(D\left[i_0,i_1,...,i_n,i_n+1,...,i_q,l\right]\right)$$
$$= \sum_{(i_n,i_{n+1},...,i_q)\in I} P_0\left(D\left[i_0,i_1,...,i_n,i_n,...,i_q\right]\right).\sum_{l=0}^{6} P_0\left(C(q+1,)\right)$$

Since the last sum equals 1, it holds:

$$\sum_{X\in M''} P_0(X) = \sum_{(i_n,i_n+1,...,i_q)\in I} P_0\left(D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q\right]\right)$$

$$\bigcup_{l=0}^{6} D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q,l\right] = D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q\right] \qquad (9)$$
$$\cap\left[\bigcup_{l=0}^{6} C(q+1,l)\right] = D\left[i_0,i_1,...,i_n,i_{n+1},...,i_q\right],$$

since the union in square brackets equals Ω by Proposition 2.4.2). This implies:

$$\bigcup\{X : X\in M''\} = \bigcup_{(i_n,_{n+1},1...,i_q)\in I}\bigcup_{l=0}^{6} D\left[i_0,i_1,...,i_n,i_{n+1},1,...,i_q,l\right]$$
$$= \bigcup_{(i_n,_{n+1},1...,i_q)\in I} D\left[i_0,i_1,...,i_n,i_{n+1},1,...,i_q,l\right].$$

So:

$$D = \bigcup\{X : X\in M\} = \bigcup\{X : X\in M'\} \cup \bigcup\{X : X\in M''\}$$
$$\bigcup\{X : X\in M'\} \cup \left(\bigcup_{(i_n,_{n+1},1...,i_q)\in I} D\left[i_0,i_1,...,i_n,i_{n+1},1,...,i_q,l\right].\right).$$

All sets in these unions are pairwise disjoint and have lengths $\le q$, so by the inductive assumption and (9):

$$P_0(D) = \sum_{X\in M'} P_0(X) + \sum_{(i_n,i_{n+1},...,i_q)\in I} P_0\left(D\left[i_0,i_1,...,_n,i_{n+1},..,i_q\right]\right)$$
$$= \sum_{X\in M'} P_0(X) + \sum_{X\in M''} P_0(X) = \sum_{X\in M} P_0(X) = \sum_{j=0}^{k} P_0(D_j).$$

This completes the proof of the inductive step. □

Lemma 2.12.

Suppose $A \in \mathscr{F}_0$ is represented as a union of pairwise disjoint planes in two ways:

$$A = \bigcup_{j=0}^{n} D_j \text{ and } A = \bigcup_{k=0}^{m} D'_k.$$

Then

$$\sum_{j=0}^{n} P_0(D_j) = \sum_{k=0}^{m} P_0(D'_k).$$

Proof. Clearly each $D_j \subseteq A$, so

$$D_j = D_j \cap A = D_j \cap \bigcup_{k=0}^{m} D'_k = \bigcup_{k=0}^{m}\left(D_j \cap D'_k\right).$$

By Proposition 2.7.4), each $D_j \cap D'_k$ is $\emptyset$ or a plane; these planes are pairwise disjoint. Since $P_0(\emptyset) = 0$, by Lemma 2.11 the following holds for any $j = 0, 1, 2,..., n$:

$$P_0(D'_k) = \sum_{j=0}^{n} P_0(D_j \cap D'_k).$$

Similarly for any $k = 0, 1, 2, ..., m$:

$$P_0(D'_k) = \sum_{j=0}^{n} P_0(D_j \cap D'_k).$$

So:

$$\sum_{j=0}^{n} P_0\left(D_j\right) = \sum_{j=0}^{n}\sum_{k=0}^{m} P_0\left(D_j \cap D'_k\right)$$
$$= \sum_{k=0}^{m}\sum_{j=0}^{n} P_0\left(D_j \cap D'_k\right) = \sum_{k=0}^{m} P_0\left(D'_k\right).$$

Definition 2.13.

Function P_0: $\mathscr{F}_0 \to [0,1]$ is defined by the following.

(1) When $A = \emptyset$ or A is a plane, $P_0(A)$ has been defined in Definition 2.10.
(2) By Proposition 2.8.2), any non-empty $A \in \mathscr{F}_0$ can be represented as $A = \bigcup_{j=0}^{m} D_j$, where D_j are pairwise disjoint planes. $P_0(A)$ is defined by:

$$P_0(A) = \sum_{j=0}^{m} P_0(D_j).$$

Due to Lemma 2.12, this definition is valid.

Lemma 2.14.

For any $m \in \mathbb{N}$ and distinct $n_0, n_1, \ldots, n_m \in \mathbb{N}$:

$$P_0\left(\bigcap_{k=0}^{m} P0(C(n_k, i_k)\right) = \prod_{k=0}^{m} P_0\left(C(n_k, i_k)\right).$$

Proof

The lemma will be proven for this particular case:

$$P_0(C(1,k) \cap c(3,l)) = P_0(C(1,k)).P_0(C(3,l)$$

The general case is proven similarly. By Proposition 2.4.2), it holds:

$$C(1,k) \cap C(3,l) = \Omega \cap C(1,k) \cap \Omega \cap C(3,l)$$
$$= \left[\bigcup_{i=0}^{6} C(0,i)\right] \cap C(1,k) \cap \left[\bigcup_{i=0}^{6} C(2,j)\right] \cap C(3,l)$$
$$= \bigcup_{i=0}^{6} \bigcup_{j=0}^{6} [C(0,i) \cap C(1,k) \cap C(2,j) \cap C(3,l] = \bigcup_{i,j=0}^{6} D[i,k,j,l].$$

So

$$P_0(C(1,k) \cap C(3,l)) = \sum_{i,j=0}^{6} P_0(D[i,k,j,l))$$
$$= \sum_{i,j=0}^{6} P_0(C(0,i).P_0(C(1,k)).P_0(C(2,j).P_0(P_0(C(3,l)$$
$$= \left[\sum_{i=0}^{6} P_0(C(0,i)\right].P_0(C(1,K)).\left[\sum_{j=0}^{6} P_0(C(2,j)\right].P_0(C(1,k)).P_0(C(3,l)$$

since each sum in square brackets equals 1 by the definition of P_0.

2.2.3. Defining P on $\mathscr{F}$.

A pre-measure P_0 on algebra $\mathscr{F}_0$ over Ω has been constructed. By Hahn–Kolmogorov extension theorem (see for example, Tao (2011)), P_0 can be extended from $\mathscr{F}_0$ to probability measure P on $\mathscr{F}$. That means:

$(\Omega, \mathscr{F}, P)$ is a probability space,
and for any $A \in \mathscr{F}_0$, $P_0(A) = P(A)$.

Lemma 2.14 implies the following corollary.

Corollary 2.15.

For any $m \in \mathbb{N}$ and distinct $n_0, n_1, \ldots, n_m \in \mathbb{N}$:

$$P\left(\bigcap_{k=0}^{m} C(n_k, i_k)\right) = \prod_{k=0}^{m} P\left(C(n_k, i_k)\right).$$

Motion of a Particle in Discrete Time and Space

Next two sections are devoted to derivation of analogues of two Newton's laws of motion from probabilistic characteristics of particle movement.

3.1. Analogue of Variance for a Random Vector

Some preliminary concepts are introduced here.

Notation 3.1.

For a random 3-dimensional vector $\boldsymbol{X} = (X_1, X_2, X_3)$ denote

$$\boldsymbol{Tr}(\boldsymbol{X}) = Var(X_1) + Var(X_2) + Var(X_3).$$

Thus, $Tr(\boldsymbol{X})$ is the trace of the covariance matrix $Cov(\boldsymbol{X})$.

Lemma 3.2.

$Tr(\boldsymbol{X}) = E[(\boldsymbol{X} - E(\boldsymbol{X}), \boldsymbol{X} - E(\boldsymbol{X}))]$, the expectation of the inner product of $\boldsymbol{X} - E(\boldsymbol{X})$ by itself.

Proof

$$\left(X - E(X), X - E(X)\right) = \sum_{i=1}^{3}\left(X_i - E(Xi)\right)^2, so$$
$$E\left[\left(X - E(X), X - E(X)\right)\right] = \sum_{i=1}^{3}, E\left[(X_i - E(X_i)\right]^2$$
$$= \sum_{i=1}^{3} Var\left(X_i\right) = TrX$$

3.2. Random Walk on a Lattice: Definitions

Notation 3.3.

1) Fix a real parameter $\tau > 0$. τ is interpreted as a unit of time. When $\tau \to 0$, time becomes continuous.
2) Fix a real number $c > 0$ and denote $\varepsilon = c\tau$.
3) Consider a 3-dimensional **lattice**:

$$\varepsilon\, \mathbb{Z}^3 = \{(\varepsilon l_1, \varepsilon l_2, \varepsilon l_3) : l_1, l_2, l_3 \in \mathbb{Z}\}.$$

When $\tau \to 0$, then $\varepsilon \to 0$ and this lattice becomes the continuous 3-dimensional space $\mathbb{R}^3$.

4) The following vectors of the standard basis in $\mathbb{R}^3$ will be used:

$$\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0) \text{ and } \mathbf{e}_3 = (0, 0, 1).$$

5) Fix the following 7 vectors:

- $\mathbf{s}(0) = (0, 0, 0)$;

- $\mathbf{s}(1) = \varepsilon\mathbf{e}_1$; $\mathbf{s}(2) = \varepsilon\mathbf{e}_2$; $\mathbf{s}(3) = \varepsilon\mathbf{e}_3$;
- $\mathbf{s}(4) = -\varepsilon\mathbf{e}_1$; $\mathbf{s}(5) = -\varepsilon\mathbf{e}_2$; $\mathbf{s}(6) = -\varepsilon\mathbf{e}_3$.

The vectors $\mathbf{s}(1)$, $\mathbf{s}(2)$, $\mathbf{s}(3)$ represent steps in the directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively, and the vectors $\mathbf{s}(4)$, $\mathbf{s}(5)$, $\mathbf{s}(6)$ represent steps in the opposite directions.

Lemma 3.4.

Suppose a random vector $\boldsymbol{X}$ has a discrete distribution with values $\mathbf{s}(0)$, $\mathbf{s}(1)$,..., $\mathbf{s}(6)$. Then $Tr(\boldsymbol{X}) \le \varepsilon^2$.
Proof. Denote $\boldsymbol{M} = E(\boldsymbol{X})$. Then

$$M = \sum_{j=0}^{6} s(j) P(X = s(j)). \tag{10}$$

$$\begin{aligned}(X, M) &= \sum_{i=1}^{3} M_i \cdot X_i \text{ and } E(X, M) = \sum_{i=1}^{3} E(M_i X_i) \\ &= \sum_{i=1}^{3}\sum_{j=0}^{6} M_i s_i(j) P(X = s(j)) \\ &= \sum_{i=1}^{3} M_i \sum_{j=0}^{6} s_i(j) P(X = s(j)) = \sum_{i=1}^{3} M_i \cdot M_i\end{aligned}$$

by (10). So:

$$E(X, M) = (M, M). \tag{11}$$

By Lemma 3.2), $Tr(\boldsymbol{X}) = E[(\boldsymbol{X} - \boldsymbol{M}, \boldsymbol{X} - \boldsymbol{M})] = E(\boldsymbol{X}, \boldsymbol{X}) - 2E(\boldsymbol{X}, \boldsymbol{M}) + (\boldsymbol{M}, \boldsymbol{M})$, since $\boldsymbol{M}$ is a constant vector. So by (11):

$$Tr(X) = E(X, X) - (M, M).$$

To complete the proof, it is sufficient to show:
$E(\boldsymbol{X}, \boldsymbol{X}) \le \varepsilon^2$:

$$\begin{aligned}E(X_1^2) &= \sum_{j=0}^{6} s_1^2(j) P(X = s(j)) = s_1^2(1) \\ &P(X = s(1)) + s_1^2(4) P(X = s(4)) \\ &= \varepsilon^2 P(X = s(1)) + \varepsilon^2 P(X = s(4)) \\ &\text{by the definition of } s(j),\ j = 0,1,2,...,6.\end{aligned}$$

$$E(X_1^2) = \varepsilon^2 \left[P(X = s(1)) + P(X = s(4)) \right]. \tag{12}$$

Similarly:

$$E(X_2^2) = \varepsilon^2 \left[P(X = s(2)) + P(X = s(5)) \right], \tag{13}$$

$$E(X_3^2) = \varepsilon^2 \left[P(X = s(3)) + P(X = s(6)) \right]. \tag{14}$$

Adding (12) - (14) produces the following:

$$E(X, X) = \sum_{i=1}^{3} E(X_i^2) = \varepsilon^2 \sum_{j=1}^{6} P(X = s(j)) \le \varepsilon^2.$$

Definitions 3.5 and 3.7 below are created by the authors as a part of the construction of models of particle movement.

Definition 3.5.

1) 3-dimensional random vectors $\mathbf{S}_n$ on $(\Omega, \mathcal{F}, P)$ are defined by the following:

$$\mathbf{S}_n(\omega) = \mathbf{s}(\omega_n),\ n \in \mathbb{N}.$$

2) A **particle walk** is defined as the sequence $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, ... of random vectors given by:

$$R_0(\omega) = s(0) = (0,0,0) \textit{ for any } \omega \in \Omega,$$

$$\textit{for } n > 1, R_n = S_0 + S_1 + ... + S_{n-1}.$$

Thus, $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, ... is a random walk on the probability space $(\Omega, \mathcal{F}, P)$. $\mathbf{R}_n$ is interpreted as the position of the particle at time $n\tau$. $\mathbf{S}_n$ is called a **step** of the particle walk; it is interpreted as displacement of the particle over the time interval $[n\tau, (n+1)\tau]$.

The particle walk $\mathbf{R}_0$, $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$, ... models movement of a particle on the lattice $\varepsilon\,\mathbb{Z}^3$: at each moment $n\tau$ in time the particle rests or moves randomly in one of the 6 directions: $\pm\,\mathbf{e}_1$, $\pm\,\mathbf{e}_2$, $\pm\,\mathbf{e}_3$. For each $\omega \in \Omega$ the sequence $(\mathbf{R}_0(\omega), \mathbf{R}_1(\omega), \mathbf{R}_2(\omega), ...)$ represents a possible path (trajectory) of the particle.

Proposition 3.6.

1) $P(\mathbf{S}_n = \boldsymbol{s}(j)) = p_n(j)$ for any $n \in \mathbb{N}$, $j \in \mathbb{N}_7$.
2) The random vectors $\mathbf{S}_0$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, ... are independent.

The first part of this proposition means: $p_n(1)$, $p_n(2)$, $p_n(3)$ are the probabilities that at time $n\tau$ the particle moves by one step in the directions $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively and $p_n(4)$, $p_n(5)$, $p_n(6)$ are the same probabilities for the directions $-\mathbf{e}_1$, $-\mathbf{e}_2$, $-\mathbf{e}_3$.

$p_n(0)$ is the probability that at time $n\tau$ the particle rests.

Proof.

$\{\mathbf{S}_n = \mathbf{s}(j)\} = \{\omega \in \Omega : \mathbf{s}(\omega_n) = \mathbf{s}(j)\} = \{\omega \in \Omega : \omega_n = j\} = C(n,j)$. Thus,

$$\{\mathbf{S}_n = \mathbf{s}(j)\} = C(n, j). \tag{15}$$

1) $P(\mathbf{S}_n = \mathbf{s}(j)) = P(C(n,j)) = p_n(j)$ by the definition of probability P.
2) For any $k > 1$, $j_0, j_1,..., j_k \in \mathbb{N}_7$ and distinct $n_1, n_2,..., n_k \in \mathbb{N}$:

$$\begin{aligned}&\{\mathbf{S}_{n1} = \mathbf{s}(j_1),...,\mathbf{S}_{nk} = \mathbf{s}(j_k)\} \\ &= \{\omega \in \Omega : \mathbf{s}(\omega_{n1}) = \mathbf{s}(j_1),...,\mathbf{s}(\omega_{nk}) = \mathbf{s}(j_k)\} \\ &= \{\omega \in \Omega : \omega_{n1} = j_1,...,\omega_{nk} = j_k\} = C(n_1, j_1) \cap ... \cap C(n_k, j_k).\end{aligned}$$

So:

$$P\left(S_{n1} = s\left(j_1\right),...,S_{n_k} = s\left(j_k\right)\right) = P\left(C\left(n_1, j_1\right) \cap ... \cap C\left(n_k, j_k\right)\right)$$
$$= P\left(C\left(n_1, j_1\right)\right) \cdot ... \cdot P\left(C\left(n_k, j_k\right)\right) = P\left(S_{n_1} = s\left(j_1\right)\right) \cdot ... \cdot P\left(S_{n_k} = s\left(j_k\right)\right).$$

The proof used Corollary 2.15 and (15).

Definition 3.7.

1) **Velocity** of the particle at time $n\tau$ is $\mathbf{v}\left(n\right) = \frac{E\left(S_n\right)}{\tau}$.

2) **Acceleration** of the particle at time $n\tau$ is

$$a\left(n\right) = \frac{v\left(n+1\right) - v\left(n\right)}{\tau}$$

This definition is used for acceleration, because acceleration is the change of velocity over unit of time.

Lemma 3.8:

$$v\left(n\right) = \frac{\varepsilon}{\tau}\sum_{i=1}^{3}\left[p_n\left(i\right) - p_n\left(i+3\right)\right]e_i.$$

Proof

Use Proposition 3.6.1) to get:

$$v\left(n\right) = \frac{1}{\tau}E\left(S_n\right)$$
$$= \frac{1}{\tau}\sum_{j=0}^{6} s\left(j\right)P\left(S_n = s\left(j\right)\right) = \frac{1}{\tau}\sum_{j=1}^{6} s\left(j\right)p_n\left(j\right)$$
$$= \frac{\varepsilon}{\tau}\sum_{i=1}^{3} p_n\left(i\right)e_i - \frac{\varepsilon}{\tau}\sum_{i=1}^{3} p_n\left(i+3\right)e_i$$
$$= \frac{\varepsilon}{\tau}\sum_{i=1}^{3}\left[p_n\left(i\right) - p_n\left(i+3\right)\right]e_i.$$

Motion with Constant Velocity

This section states an axiom about probabilities of particle motion under zero resultant force; then an analogue of Newton's first law of motion is derived.

Axiom 1. For any $n \in \mathbb{N}$, if no force acts on the particle during time $[n\tau, (n+1)\tau]$, then $p_{n+1}(j) = p_n(j)$ for any $j \in \mathbb{N}_7$.

In other words, this axiom states that with no force acting on the particle the probabilities of its movement in each direction stay constant.

Instead of Axiom 1 its weaker form - Axiom 1* will be used.

Axiom 1*. For any $n \in \mathbb{N}$, if the resultant force on the particle during time interval $[n\tau, (n+1)\tau]$ equals zero, then $p_{n+1}(i) - p_{n+1}(i+3) = p_n(\mathrm{i}) - p_n(\mathrm{i}+3)$ for each $i \in \{1,2,3\}$.

This axiom refers to a zero resultant force, because it is well-known that motion under zero resultant force is the same as under no force. Clearly, Axiom 1* follows from Axiom 1. Due to Lemma 3.8, Axiom 1* implies that with zero resultant force the particle's velocity is constant during the interval.

Theorem 4.1.

Suppose the resultant force on the particle over time interval $[0, N\tau]$ equals zero ($N \in \mathbb{N}$). Then the following hold:

1) The particle moves with constant velocity during the time interval $[0, N\tau]$, that is the velocity $\mathbf{v} = \mathbf{v}(\mathrm{n})$ is the same for all $n \leq N$.
2) For any $n \leq N$:

$$E(\mathbf{R}_n) = n\tau\mathbf{v}.$$

3) Suppose $N \to \infty$ in such a way that $N\tau$ = Const (so the time interval $[0, N\tau]$ is fixed). Then

$$Tr(\mathbf{R}_N) \to 0.$$

This is interpretation of Theorem 4.1: when no force acts on the particle during a large time interval, then:

- the particle's motion follows approximately the Newton's first law of motion during this time interval: the motion is uniform and is along a straight line (parts 1) and 2) of the theorem);
- the deviation from this law gets very small as the time unit τ approaches zero, i.e. time becomes continuous (part 3) of the theorem). The time unit $\tau = \frac{Const}{N}$, so $\tau \to 0$ as $N \to \infty$.

Proof

1) By Lemma 3.8, $v\left(n\right) = \frac{3}{\tau}\sum_{i=1}^{3}\left[p_n\left(i\right) - p_n\left(i+3\right)\right]e_i$.

Denote $a_i = p_n(i) - p_n(i+3)$. By Axiom 1*, a_i does not depend on $n \in [0,N]$ for any $i \in \{1, 2, 3\}$. So

$v\left(n\right) = \frac{3}{\tau}\frac{3}{i=1}a_i e_i$ is the same for all $n \leq N$.

2) Suppose $n \leq N$. By the definition of velocity, $E(\mathbf{S}_n) = \tau\mathbf{v}(n) = \tau\mathbf{v}$ by part 1). $\mathbf{R}_n = \mathbf{S}_0 + \mathbf{S}_1 + ... + \mathbf{S}_{n-1}$, so

$$E\left(R_n\right) = \sum_{i=0}^{n-1}E\left(S_i\right) = \sum_{i=0}^{n-1}\tau v = n\tau v.$$

3) Suppose $N \to \infty$ in such a way that $N\tau = C$, where C is a constant.

By Lemma 3.4, $Tr(\mathbf{S}_n) \leq \varepsilon^2$ for any $n \in \mathbb{N}$. By Proposition 3.6.2), the random vectors $\mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_3,...$ are independent. So

$$Tr\left(R_n\right) = \sum_{i=0}^{n-1} Tr\left(S_i\right) \leq n\varepsilon^2 \leq N\varepsilon^2.$$

Since $\varepsilon = c\tau = \frac{cC}{N}$, it holds that:

$$Tr\left(R_n\right) \leq N\varepsilon^2 \leq N\left(\frac{cC}{N}\right)^2 = \frac{(cC)^2}{N} \to 0\, as\, N \to \infty$$

Accelerated Motion

This section considers motion of the particle under a non-zero force.

Proposition 5.1. (Recurrence relation).

Motion of the particle under any force satisfies the following recurrence relation:

$$E\left(R_{n+2}\right) = 2E\left(R_{n+1}\right) - E\left(R_n\right) + \tau^2 \mathrm{a}\left(n\right)$$

This is an analogue of the approximate formula for acceleration:

$$\ddot{r} \approx \frac{r(t+\tau) - 2r(t) + r(t-\tau)}{\tau^2},$$

where $\mathbf{r}(t)$ is the position function; this formula is easily derived using Taylor's formula.

Proof

By the definitions of velocity and $\mathbf{R}_n$ it holds:

$$v(n) = \frac{1}{\tau} E\left(S_n\right) = \frac{1}{\tau}\left[E\left(E_{n+1}\right) - E\left(R_n\right)\right],$$
$$v(n+1) = \frac{1}{\tau}\left[E\left(R_{n+2}\right) - E\left(R_{n+1}\right)\right].$$

Subtracting these equalities produces the following:

$$a(n) = \frac{1}{\tau}\left[v(n+1) - v(n)\right] = \frac{1}{v^2}\left[E\left(R_{n+2}\right) - 2E\left(R_{n+1}\right) + E\left(R_n\right)\right]$$

And:

$$E\left(R_{n+2}\right) - 2E\left(R_{n+1}\right) + E\left(R_n\right) = \tau^2 a(n).$$

Next results are derived from the following axiom about probabilities of the particle movement.

Axiom 2. *Let* $\boldsymbol{F}(n) = (F_1(n), F_2(n), F_3(n))$ be a resultant force acting on the particle at time $n\tau$ and let each $F_i(n)$ be a difference of two forces:

$$F_i(n) = f_n(i) - f_n(i+3),$$

where the force $f_n(i)$ acts in the direction of $\mathbf{e}_i$ and $f_n(i+3)$ - in the opposite direction. Assume that there is a constant $\gamma > 0$ such that for any $j = 1, 2, 3, 4, 5, 6$ and $n \leq N$:

$$p_{n+1}(j) - p_n(j) = \gamma f_n(j).$$

Axiom 2 means that during each time unit the change in the probability of moving in direction j is proportional to the force acting in this direction. In short: changes in motion probabilities are proportional to the forces acting on the particle.

Theorem 5.2.

The resultant force acting on the particle is proportional to its acceleration, i.e. there is a constant $\beta > 0$ such that for any $n \leq N$:

$$F(n) = \beta a(n). \tag{16}$$

This theorem is a weaker analogue of Newton's second law of motion: $\mathbf{F} = m\mathbf{a}$. From (16) one can define the **mass** of the particle as the coefficient of proportionality β.

Proof

For any $n \leq N$ and $i \in \{1, 2, 3\}$ by Axiom 2 it holds:

$$F_i(n) = f_n(i) - f_n(i+3) = \frac{1}{\gamma}$$
$$\left[p_{n+1}(i) - p_n(i)\right] - \frac{1}{\gamma}\left[p_{n+1}(i+3) - p_n(i+3)\right]$$
$$= \frac{1}{\gamma}\left[p_{n+1}(i) - p_{n+1}(i+3)\right] - \frac{1}{\gamma}\left[p_n(i) p_n(i+3)\right].$$

So

$$F(n) = \sum_{i=1}^{3} F_i(n) e_i$$
$$= \frac{1}{\gamma}\sum_{i=1}^{3}\left[p_{n+1}(i) - p_{n+1}(i+3)\right]e_i - \frac{1}{\gamma}$$
$$\sum_{i=1}^{3}\left[p_n(i) - p_n(i+3)\right]e_i$$
$$= \frac{1}{\gamma}\cdot\frac{\tau}{\varepsilon}v(n+1) - \frac{1}{\gamma}$$

by Lemma 3.8 and the definition of acceleration $\mathbf{a}(n)$.

Thus, $\boldsymbol{F}(n) = \beta\mathbf{a}(n)$, where $\beta = \dfrac{\tau^2}{\varepsilon\gamma} > 0$.

Proposition 5.3. (Motion under constant forces).

Suppose in each direction the acting force is constant, that is for any j = 1, 2, ..., 6, $f(j) = f_n(j)$ does not depend on n ($n \le N$). Then there exist constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that for any $n \le N$:

$$E(R_n) = \varepsilon n\left(\frac{n-1}{2}a + b\right) \tag{17}$$

Proposition 5.3 means that when the particle moves under constant forces, its trajectory is approximately a parabola in 3-D. This is because (17) is a parametric equation of a parabola (with quadratic function of time $n\tau$).

Proof

By Axiom 2 it holds:

$$p_{n+1}(j) = p_n(j) + \gamma f(j). \tag{18}$$

Denote

- $a_i = \gamma[f(i) - f(i+3)]$, $b_i = p_0(i) - p_0(i+3)$,
- $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$.

The following formula will be proven by induction on n:

$$p_n(i) - p_n(i+3) = a_i n + b_i. \tag{19}$$

If $n = 0$, then the formula obviously holds.
Assume it holds for n. By (18),

$$\begin{aligned}
&p_{n+1}(i) - p_{n+1}(i+3)\\
&= \left[p_n(i) + \gamma f(i)\right]\\
&-[p_n(i+3) + \gamma f(i+3)]\\
&= p_n(i) - p_n(i+3) + \gamma\left[f(i) - f(i+3)\right]\\
&= (a_i n + b_i) + a_i = a_i(n+1) + b_i
\end{aligned}$$

by the inductive assumption. This completes the proof of (19).
By Lemma 3.8 and (19):

$$E(S_n) = \tau v_n = \varepsilon\sum_{i=1}^{3}\left[p_n(i)\,p_n(i+3)\right]e_i$$

$$E(S_n) = \varepsilon\sum_{i=1}^{3}(a_i n + b_i)e_i \tag{20}$$

$$E(R_n) = E(S_0) + E(S_1) + \ldots + E(S_{n-1}).$$

By (20) the first coordinate of $E(\mathbf{S}_n)$ is $E(\mathbf{S}_n)_1 = \varepsilon(a_1 n + b_1)$, so the first coordinate of $E(\mathbf{R}_n)$ equals:
$E(\mathbf{R}_n)_1 = \varepsilon b_1 + \varepsilon(a_1+b_1) + \varepsilon(2a_1+b_1) + \ldots + \varepsilon[a_1(n-1)+b_1]$

$= \varepsilon[nb_1 + a_1(1 + 2 + \ldots + (n-1))]$ and

$$E(R_n)_1 = \varepsilon n\left[\frac{n-1}{2}a_1 + b_1\right] \tag{21}$$

Similarly, for the second and third coordinates of $E(\mathbf{R}_n)$ it holds:

$$E(R_n)_2 = \varepsilon n\left(\frac{n-1}{2}a_2 + b_2\right) \tag{22}$$

$$E(R_n)_3 = \varepsilon n\left(\frac{n-1}{2}a_3 + b_3\right) \tag{23}$$

Combining (21) - (23) produces

$$E(R_n) = \varepsilon n\left(\frac{n-1}{2}a + b\right)$$

Conclusion

To summarize the contribution of this paper: here the authors use a rigorous approach to construct a probability space for trajectories of a particle on three-dimensional lattice and they create a model of particle motion as a random walk on this probability space. Based on two natural assumptions about probabilistic characteristics of particle movement, they derive analogues of Newton's first and second laws of motion, as well as other related facts. While in previous research these Newton's laws were considered axioms, in this paper they are derived from axioms about particle movement. This probabilistic approach is more appropriate for non-equilibrium mechanics. It should simplify derivation of ergodic properties of many-particle systems and potentially lead to derivation of other classical properties of time-dependent systems.

As described in the Introduction, a similar approach can be applied to resolve some inconsistencies in thermodynamics theory. Next step in this direction of research is generalizing the probabilistic model developed in this paper to many-particle systems. The aim is to use such a model to formally derive laws of thermodynamics in a consistent way.

Acknowledgment

The authors thank the reviewer for their valuable comments that helped to improve this study.

Author's Contributions

Farida Kachapova: Is responsible for formal analysis, supervision, writing, and reviewing.

Ilias Kachapov: Is responsible for conceptualization, methodology, writing, and proofs.

Ethics

This is a mathematical article; no ethical issues can arise after its publication.

References

Attard, P. (2012). Non-equilibrium thermodynamics and statistical mechanics: Foundations and applications. OUP Oxford. ISBN-10: 019163977X.

Billingsley, P. (2012). Probability and measure, anniversary ed. ISBN-10: 978-1-118-12237-2.

Livi, R., & Politi, P. (2017). Nonequilibrium statistical physics: A modern perspective. Cambridge University Press. ISBN-10: 9781107049543.

Lykov, A., & Malyshev, V. (2018). Convergence to equilibrium due to collisions with external particles. arXiv preprint arXiv:1812.08242. https://arxiv.org/abs/1812.08242

Ropke, G. (2013). Nonequilibrium statistical physics, Wiley, 2013. ISBN-10: 9783527410927.

Tao, T. (2011). An introduction to measure theory (Vol. 126, p. 206). Providence: American Mathematical Society. ISBN-10: 9780821869192.