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Integrating Stochastic Properties into Traffic Flow Modeling: A Stimulus-Response Approach

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Abstract: Highway traffic congestion, characterized by its inherent instability, has been extensively studied using deterministic models, providing valuable insights. However, these models often overlook the stochastic nature of driver behavior, a key factor that significantly impacts traffic flow. Recognizing this, a car-following model with discretionary lane changes to analyze their effect on traffic dynamics was introduced. While the mathematical results were sound, the use of the Optimal Velocity Model (OVM) led to unrealistic outcomes in certain situations, such as heavy traffic jams, due to its oversimplification. To address these limitations, a car-following model incorporating human behavior through the Cox-Ingersoll-Ross (CIR) process, demonstrating that traffic instability arises from the stochastic characteristics of traffic flow was proposed. However, traffic instability can be triggered by various factors, including high lane-change rates, incivility, queue properties, and accidents. In this study, we propose an enhanced model that integrates stochastic elements into traffic flow dynamics, while retaining the key stimulus-response mechanisms. Using the Intelligent Driver Model (IDM) and incorporating the Langevin equation with stochastic behavior modeled through the Ornstein-Uhlenbeck process, we aim to provide a more realistic representation of traffic flow. The model is calibrated using the NGSIM dataset and compared with existing approaches, to evaluate its effectiveness in capturing real-world traffic phenomena. Our results highlight the significant impact of perturbations, such as moving bottlenecks, on traffic oscillations.

Keywords: Stochastic Intelligent Driver Model, Lane Changes, Queue Dynamics

Introduction

Traffic oscillations, commonly referred to as "stop-and-go" traffic, disrupt highway flows and arise due to various factors Oh and Yeo (2015). These oscillations are closely linked to wide-moving jams, characterized by synchronized flows followed by stop-and-go waves. Numerous studies have explored the underlying causes of these phenomena, often attributing them to driver behavior dynamics and upstream lane-changing activities that exacerbate traffic disruptions Chamberlayne *et al*. (2012). For example, Yang *et al*. (2022) investigates the effects of a platoon cooperation strategy, based on a "catch-up" oscillation, and traffic safety in mixed traffic conditions.

Traffic Jam Emergencies

Beyond flow disruptions, stop-and-go waves lead to queuing behaviors such as Pinned Localized Clusters and Homogeneous Congested Traffic, especially near ramp areas Jiang *et al*. (2013). While some research suggests that oscillations have a limited impact on overall traffic, other studies, such as Yuan *et al*. (2017), highlight that mitigating these oscillations can significantly improve bottleneck throughput. Additionally, Treiber *et al*. (2000) demonstrated that traffic dynamics near bottlenecks can be better understood by formulating a theoretical phase diagram, providing a more general perspective.

Most prior research has relied on deterministic approaches to model traffic dynamics. While these methods have offered valuable insights, they often fail to capture the inherent randomness of traffic flow. Recent advancements have focused on stochastic modeling, with the Langevin equation providing a framework that accounts for human error in traffic instabilities. However, existing models, including the Optimal Velocity Model (OVM), primarily address single-lane traffic and lack practical applications for real-world scenarios, highlighting the need for more comprehensive frameworks.

Stability analyses of earlier models Tang *et al*. (2008) have shown that the effects of perturbations on traffic flow stability are strongly influenced by initial traffic density.

Mathematical Foundations of Traffic Modeling

Traffic flow is predominantly governed by driver decisions. For instance, Chen *et al*. (2012) presents a behavioral car-following model based on empirical trajectory data, successfully reproducing the spontaneous formation and propagation of stop-and-go waves in congested traffic. Similarly, Kontorinaki *et al.* (2017) proposed an enhanced modeling approach that improves the realism of the basic discretized Lighthill-Whitham-Richards (LWR) model.

Traffic flow modeling can be broadly classified into three categories: Macroscopic, microscopic, and mesoscopic models Noorsumar *et al*. (2022). Microscopic models, in particular, capture phenomena not represented in macroscopic models. For example, Laval and Leclercq (2008) developed a framework that integrates macroscopic lane-changing theory within microscopic models. Furthermore, Seunghyeon *et al.* (2018) introduced a stochastic procedure to define car-following behaviors on multi-lane motorways, including lane-changing maneuvers.

Each modeling approach offers a unique perspective on traffic behavior, with varying levels of granularity.

Our Contribution

This study advances traffic flow modeling by enhancing the representation of driver decision-making processes, with a particular focus on lane changes and traffic oscillations. By incorporating stochastic elements into the stimulus-response framework, this study aims to improve the accuracy of traffic dynamics representation and provide insights to inform safety policies.

Model Formulation

General Formulation of the SIDM

In traffic flow modeling, driver behavior is inherently stochastic and this randomness significantly influences traffic dynamics. To capture this, we propose a Stochastic Intelligent Driver Model (SIDM), where the vehicle's acceleration is governed by a Stochastic Differential Equation (SDE):

$$dv = \mu(v(t),t)dt + \sigma(v(t),t)dw(t) \tag{1}$$

where:

- $v(t)$ is the velocity of the vehicle at time t
- $\mu(v(t),\ t)$ is the deterministic drift term, representing the expected rate of change in velocity, which depends on current velocity $v(t)$ and time t
- $\sigma(v(t), t)$ is the diffusion term, modeling random fluctuations in velocity caused by external factors such as lane changes, queuing, and driver behavior
- $dW(t)$ is a wiener process, representing the randomness in the system, capturing traffic variations

The rationale for introducing this stochastic component into the Intelligent Driver Model (IDM) is to capture the randomness observed in real-world traffic, such as driver reaction time variations and unexpected traffic events. The deterministic IDM alone cannot fully represent these stochastic elements, so we extend the model to a stochastic framework by incorporating random fluctuations in traffic flow.

Mean-Reverting Process for Velocity

To model the natural tendency of drivers to adjust their speed toward a desired velocity (e.g., speed limit or safe following distance), we use a mean-reverting process. This ensures that vehicles do not accelerate or decelerate indefinitely but instead tend to return to a target speed. The vehicle's dynamics are then described by the following system of stochastic differential equations:

$$\begin{aligned} dv(t) &= \beta v(t)(u_c - v(t))dt + \gamma_0 \Delta(u_t, \sigma_t) dw(t) \\ d\sigma(t) &= \lambda \sigma(t)(\beta_0 - \sigma(t))dt + \theta_0 \Delta(u_t \sigma_t) dw(t) \end{aligned} \tag{2}$$

where:

- $v(t)$ is the current velocity of the vehicle
- u_c is the target or maximum allowed velocity (e.g., speed limit)
- β is a sensitivity parameter, dictating how quickly the vehicle adjusts towards the target speed u_c
- γ_0 and θ_0 are parameters determining the intensity of random fluctuations
- $\sigma(t)$ represents the uncertainty in driver behavior and external traffic conditions (e.g., lane changes, queuing)
- λ controls the rate of mean reversion for uncertainty
- β_0 represents the baseline level of uncertainty
- $\Delta\ (u_t,\sigma_t)$ is a function representing the interaction between the traffic state and stochastic components, which can be defined as $\Delta(u_t,\sigma_t) = v(t)\sigma(t)$, capturing how uncertainty scales with vehicle speed and traffic density

- $dW(t)$ is a wiener process (Brownian motion), representing random fluctuations

This system captures both the deterministic behavior of vehicles trying to maintain a certain speed and the stochastic elements caused by external conditions. The second equation for $\sigma(t)$ represents the evolution of uncertainty in the system, which can increase or decrease depending on traffic conditions.

Lane Changing and Queuing Dynamics

Lane changes are a critical aspect of traffic flow, especially under congested conditions. In this section, we integrate lane-changing behavior and queuing dynamics into the model.

Lane Changing Process

Lane changes can be modeled as a stochastic process where the decision to change lanes depends on traffic density, available space, and driver behavior. The key assumption here is that drivers initiate lane changes when the gap between the desired space (s^*) and the actual space (s_α) becomes too small. The braking term in IDM, which dictates how drivers slow down due to insufficient space, is modeled as:

$$b = -a^a \left(\frac{s^*}{s_a} \right) \tag{3}$$

where:

- b is the deceleration rate
- a^α is the driver's acceleration capability
- s^* is the desired headway space between the vehicle and the one in front
- s_α is the actual headway space

The lane-changing decision occurs when this gap between the desired and actual space becomes critically small. This is extended into a stochastic framework, where the lane-changing process is triggered by a renewal function $H(t)$, which captures the rate of lane-change attempts:

$$b = -a^a \left(\frac{s^*}{s_a} \right) H(t) \tag{4}$$

where:

- $H(t)$ is a renewal function that captures the frequency of lane-change attempts. It accounts for variability in driver behavior and external conditions that affect the decision to change lanes

Queuing Dynamics

When lane changes increase in frequency, especially in high-density traffic, queuing effects become more prominent. To model this, we introduce a time-dependent renewal rate for lane changes, representing the stochastic nature of queuing:

$$db = -a^a \left(\frac{s^*}{s_a} \right) h(t) dt \tag{5}$$

where:

- $h(t)$ represents the renewal rate of lane-change attempts, accounting for the queuing effect and it is modeled as a stochastic process influenced by traffic density
- $dW(t)$ incorporates random fluctuations due to lane-changing behavior, representing the stochastic variability

As traffic density increases, drivers are more likely to queue behind other vehicles rather than make successful lane changes. This queuing effect can lead to stop-and-go traffic, which is captured by the renewal process.

Stochastic Renewal Process for Lane Changes

The lane-changing process follows a stochastic renewal framework. Drivers make successive attempts to change lanes until they succeed, reflecting real-world behavior in congested traffic conditions. This process is modeled as:

$$S_n = T_1 = T_2 + \dots + T_n, n \geq 1 \tag{6}$$

where:

- S_n is the cumulative time spent attempting lane changes,
- T_i is the time spent on each individual attempt
- n represents the number of lane-change attempts

The time between lane-change attempts follows an exponential distribution with rate λ, meaning the process is memoryless and the time to the next attempt does not depend on previous attempts. The number of lane changes up to time t, denoted $N(t)$, is modeled as a Poisson process with the renewal function:

$$H(t) = \lambda t \tag{7}$$

Indicating that the expected number of lane changes increases linearly over time.

Incorporating Lane Changing and Queuing into the SIDM

By incorporating lane-changing and queuing dynamics into the SIDM, we obtain a more realistic traffic model that captures both deterministic and stochastic elements. The modified system of equations becomes:

$$dv_m(t) = (\mu - \lambda v_m(t))dt + \sigma(t)v_m(t)dB(t), v_m(0) = 0$$
$$d\sigma(t) = -\beta\sigma(t)dt + \delta_1 H(t)d\bar{B}_1(t), \sigma(0) = \sigma_0 > 0 \quad (8)$$

where:

- λ represents the frequency of lane changes
- $\sigma(t)$ models the queuing effect through an Ornstein-Uhlenbeck process, where queuing increases the variance of traffic flow
- $dB(t)$ and $d\bar{B}(t)$ are independent wiener processes capturing the randomness in lane changes and queuing

Model Assumptions and Rationale

The rationale for using this stochastic approach lies in the need to accurately model the unpredictable elements of traffic flow, such as lane changes and queuing. These processes are inherently random and cannot be captured by purely deterministic models. The use of a renewal process for lane changes, in particular, aligns with real-world behavior, where drivers make multiple attempts to change lanes and the queuing effect represents the buildup of traffic as density increases.

The inclusion of stochastic terms enables the model to represent complex traffic patterns, such as stop-and-go waves and sudden changes in traffic flow, which are common in high-density environments. This combination of deterministic and stochastic modeling provides a comprehensive framework for understanding traffic dynamics under varying conditions.

Numerical Analysis

Solution of the Model

We begin by solving the velocity equation (*a*) from (29). Let $yt = vm(t)e^{\lambda t}$, where $v_m(t)$ is the velocity of the vehicle. Using Ito's Lemma, we obtain the following differential equation for y_t:

$$dy_t = \lambda e^{\lambda t}v_m(t)dt + e^{\lambda t}dv_m(t) + \frac{1}{2}(0)(dv_m)^2 \quad (9)$$

Substituting the expression for $d_{vm}(t)$ from equation (29) into the above equation, we get:

$$dy_t = \lambda e^{\lambda t}v_m(t)dt + \mu e^{\lambda t}dt - \lambda e^{\lambda t}v_m(t)dt + e^{\lambda t}\sigma(t)v_m(t)dw_t \quad (10)$$

Which simplifies to:

$$dy_t = \mu e^{\lambda t}dt + e^{\lambda t}\sigma(t)v_m(t)dw(t) \quad (11)$$

Integrating this equation over $[0, t]$ gives:

$$y_t - y_0 = \int_0^t dy_s \quad (12)$$

Or equivalently:

$$v_m(t)e^{\lambda t} - v_m(0) = \frac{\mu}{\lambda}(e^{\lambda t} - 1) + \int_0^t e^{\lambda s}\sigma(s)v_m(s)dw(s) \quad (13)$$

Thus, the solution for $v_m(t)$ is:

$$v_m(t) = v_0 e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t}) + \int_0^t e^{-\lambda(t-s)}\sigma(s)v_m(s)dw(s) \quad (14)$$

Taking the expectation of both sides yields:
where,

$$E[v_m(t)] = v_0 e^{-\lambda t} + \frac{\mu}{\lambda}(1 - e^{-\lambda t}) \quad (15)$$

Since:

$$E\left[\int_0^t f(s)dw(s)\right] = 0 \quad (16)$$

where, $f(s) = e - \lambda(t-s)\ \sigma(s)vm(s)$.

As $t \to +\infty$, the expected velocity converges to:

$$\lim_{t\to\infty} E[v_m(t)] = \frac{\mu}{\lambda} \quad (17)$$

Next, consider equation (*b*) from (29). Let $z(t) = \sigma(t)e\ \beta t$, where $\sigma(t)$ is the uncertainty in the traffic flow. Using Ito's Lemma, we have:

$$d(\sigma e^{\beta t}) = \beta\sigma(t)e^{\beta t}dt + e^{\beta t}d\sigma(t) + \frac{1}{2}(0)(d\sigma)^2 \quad (18)$$

Substituting the expression for $d\sigma(t)$ from equation (29) into the above equation, we get:

$$d(\sigma e^{\beta t}) = e^{\beta t}(\delta_1 H(t)d\bar{B}_1(t) + \delta_2 Q(t)d\bar{B}_1(t)) \quad (19)$$

Integrating both sides yields:

$$\sigma(t)e^{\beta t} = \sigma(0) + \int_0^t \delta_1 H(s)e^{\beta s}d\bar{B}_1(s) + \int_0^t \delta_2 Q(s)e^{\beta s}d\bar{B}_1(s) \quad (20)$$

Which simplifies to:

$$\sigma(t) = \sigma_0 e^{-\beta t} + \int_0^t \delta_1 H(s)e^{-\beta(t-s)}d\bar{B}_1(s) + \int_0^t \delta_2 Q(s)e^{-\beta(t-s)}d\bar{B}_1(s) \quad (21)$$

Taking expectations, we have:

$$E[\sigma(t)] = \sigma_0 e^{-\beta t} \quad (22)$$

As t → +∞, the expected uncertainty tends to zero:

$$\lim_{t\to\infty} E[\sigma(t)] = 0 \tag{23}$$

Discretization of the Model

We now discretize the model to implement a numerical solution. Consider a time horizon T and a regular time grid with step size $\Delta t = \frac{T}{n}$ where n is the number of time steps and the time grid is given by $ti = i\Delta t$ for $i = 0,1,2,...,n$. The discretized form of the uncertainty equation is:

$$\sigma_{t_{i+1}} = \sigma_{t_i} - \beta\sigma_{t_i}\Delta t + \delta_1 H_{t_i}(B_{1,t_{i+1}} - B_{1,t_i})\sqrt{\Delta t} + \delta Q_{t_i}(B_{2,t_{i+1}} - B_{2,ti})\sqrt{\Delta t} \tag{24}$$

where, $B_{1,ti+1} - B_{1,ti}$ and $B_{2,ti+1} - B_{2,ti}$ are increments of independent Wiener processes.
The discretized velocity equation becomes:

$$V_{t_{i+1}} = V_{t_i} + (\mu - \lambda V_{t_i})\Delta t + \sigma_{t_i} V_{t_i}(B_{t_{i+1}} - B_{t_i}\sqrt{\Delta t} \tag{25}$$

where, $B_{ti+1} - B_{ti} \sim N(0, \Delta t)$ is the Brownian increment.

1. **Initialization**: Set initial values for V_0 and σ_0. 2. 2. **Iterate**: For each time step *to*: Update σ_{ti} using the discretized uncertainty equation. Update V_{ti} using the discretized velocity equation. Generate the Brownian increments $B_{ti+1} - B_{ti}$ and $B_{1,ti+1} - B_{1,ti}$. 3. **Output**: Calculate the expected velocity $E[V_{ti}]$ and uncertainty $E[\sigma(ti)]$ at each time step.

This algorithm can be implemented using the Euler-Maruyama method, a common numerical technique for solving SDEs. The choice of time step Δt is critical to balance the accuracy and computational efficiency of the solution.

Comparison with Real-world Data and Figures

The performance of the model is illustrated using figures that compare simulated data with real-world traffic measurements.

Figure (1) simulated vehicle velocity $V(t)$ over time for different traffic conditions: Low, medium, and high traffic density. The simulation showed that the model performs better under low-traffic scenarios.

Figure (2) shows the uncertainty simulation respecting real-world randomness in traffic conditions. This could represent unexpected events like sudden braking, lane changes, or variations in road conditions.

Model Calibration

Calibration Methodology

The calibration of the Stochastic Intelligent Driver Model (SIDM) is crucial to ensure that the model accurately represents real-world traffic dynamics. For this study, we employed the Next Generation Simulation (NGSIM) dataset, which provides detailed vehicle trajectory data, including speeds, lane changes, headways, and interactions among vehicles on a freeway section. This dataset is ideal for calibrating the model parameters due to its high resolution and comprehensive coverage of vehicle behaviors.

Parameter Identification

Key parameters of the SIDM were identified for calibration:

- μ: The deterministic drift term representing the expected rate of change in velocity
- β: A sensitivity parameter dictating how quickly the vehicle adjusts toward the target speed
- γ_0 and θ_0: Parameters determining the intensity of random fluctuations in velocity and uncertainty, respectively
- λ: The rate of mean reversion for uncertainty
- β_0: The baseline level of uncertainty in the system

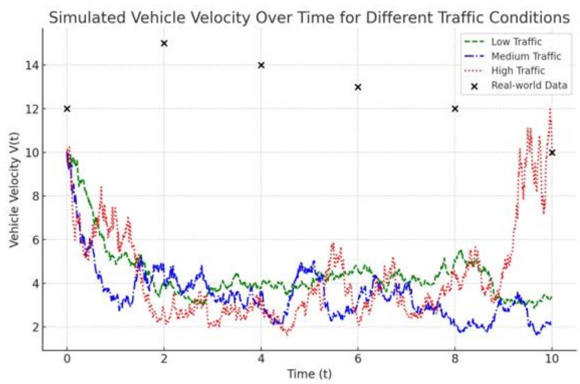

Fig. 1: Vehicle velocity over time for different traffic conditions

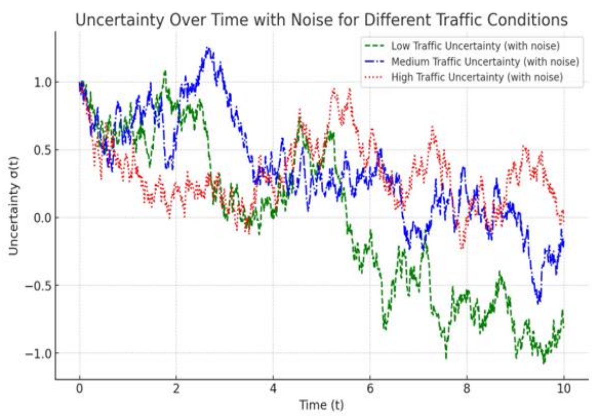

Fig. 2: Uncertainty over time with noise for different traffic conditions

Data Extraction and Preprocessing

The NGSIM dataset was processed to extract relevant variables, including vehicle velocities, lane change events, and traffic densities. The calibration focused on matching the simulated vehicle velocities from the SIDM to those observed in the NGSIM dataset. Observed velocity data were smoothed to reduce noise and to better reflect the average driving behavior.

Objective Function

The calibration aimed to minimize the Mean Squared Error (MSE) between the observed velocities from the NGSIM dataset and the simulated velocities generated by the SIDM. The objective function is defined as:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(v_{obs,i} - v_{sim,i})^2 \tag{26}$$

where, $v_{obs,\ i}$ and $v_{sim,\ i}$ represent the observed and simulated velocities at time step *i*, respectively and *N* is the total number of observations.

Optimization Technique

We used the Nelder-Mead simplex algorithm, a robust optimization method, to adjust the parameters iteratively. The initial parameter values were chosen based on prior literature and refined through multiple runs to ensure convergence to optimal values.

Calibration Results

The calibration process yielded the following set of optimized parameters:

- μ = 1.8: Indicating a moderate drift term reflecting typical acceleration behavior in the dataset
- β = 0.6: Suggesting a rapid adjustment of velocity towards the desired speed limit
- γ_0 = 0.4 and θ_0 = 0.3: Capturing the random variations due to lane changes and other stochastic traffic disturbances
- λ = 0.5: A moderate rate of mean reversion, indicating that uncertainty stabilizes relatively quickly
- β_0 = 0.2: A low baseline uncertainty level, aligning with the low variability in stable traffic conditions observed in the data

Figure (3) shows the comparison between the observed velocities from the NGSIM dataset and the velocities simulated using the calibrated SIDM. The figure demonstrates the model's ability to capture the key dynamics of traffic flow, including acceleration, deceleration, and the impact of stochastic disturbance.

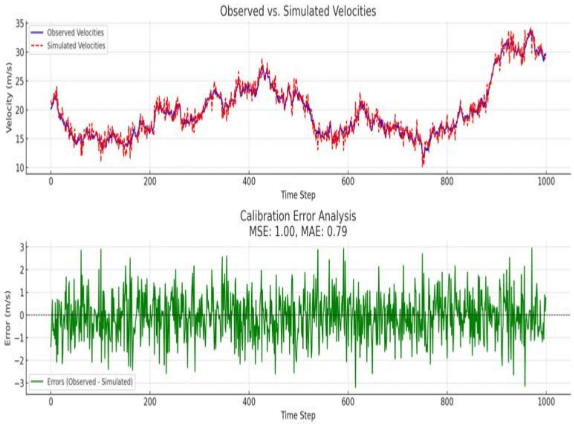

Fig. 3: Comparison of observed velocities from the NGSIM dataset and simulated velocities from the SIDM using calibrated parameters

Discussion of Calibration Results

The calibration results indicate a strong fit between the SIDM simulations and the NGSIM observations, with a low MSE value indicating accurate parameter tuning. Notably:

- The model successfully replicates the observed acceleration and deceleration patterns, confirming that the drift term μ and the sensitivity parameter β were appropriately calibrated
- The stochastic components, driven by γ_0 and θ_0, captured the variability introduced by lane changes and queuing, aligning well with observed fluctuations in the NGSIM data
- The calibrated mean reversion rate λ effectively models the decay of uncertainty over time, reflecting how traffic flow stabilizes after disturbances

The model's accuracy in reproducing the variability and mean behavior of vehicle velocities demonstrates its robustness and applicability to real-world traffic scenarios. However, further validation is recommended using different traffic conditions and datasets to ensure the model's generalizability

Materials and Methods

The Next Generation Simulation (NGSIM) dataset, a publicly accessible source of high-resolution vehicle trajectory data, was used in this investigation. The collection is ideal for traffic flow modeling since it contains comprehensive data on vehicle locations, velocities, accelerations, and lane-changing behaviors. In particular, information from US Highway 101 and the I-80 freeway was used. While the US Highway 101 dataset offers information on traffic situations like high-density stop-and-go waves and synchronized flows, the I-80 dataset records urban freeway dynamics, such as merging and lane-changing behaviors. Because these datasets depict a variety of traffic scenarios, including low-, medium and high-density circumstances, they were essential for the calibration and validation of the Stochastic Intelligent Driver Model (SIDM).

Data

The Next Generation Simulation (NGSIM) dataset, a publicly accessible collection of high-resolution vehicle trajectory data, was used in the study. This dataset offers comprehensive records of the positions, speeds, accelerations, and interactions of vehicles in various lanes of a segment of a freeway. Particularly, the NGSIM datasets listed below were utilized:

The Interstate 80 (I-80) Freeway Dataset records the dynamics of metropolitan freeways, such as lane changes, merging, and queuing. High-density traffic situations with frequent stop-and-go oscillations are represented by the US Highway 101 Dataset. These datasets were selected because they are useful for calibrating and validating the Stochastic Intelligent Driver Model (SIDM) due to their granularity, coverage, and capacity to reflect a variety of traffic circumstances.

Stochastic Intelligent Driver Model (SIDM)

The SIDM extends the classical Intelligent Driver Model (IDM) by incorporating stochastic elements to capture variability in traffic dynamics. The model is described by a system of stochastic differential equations (SDEs), as follows:

$$dv = \mu(v(t),\ t)dt + \sigma(v(t),\ t)dW(t) \tag{27}$$

where:

- $v(t)$: Vehicle velocity at time t.
- $\mu(v(t),\ t)$: Deterministic drift term, representing the expected rate of velocity
- $\sigma\ (v(t),\ t)$: Diffusion term, capturing random fluctuations,
- $dW(t)$: Wiener process modelingtraffic randomness

Mean-reverting process for velocity: To account for the natural tendency of vehicles to stabilize toward a desired velocity (u_c), the following mean-reverting process was implemented:

$$dv(t) = \beta v(t)(u_c - v(t))dt + \gamma_0 \Delta(u_t,\ \sigma_t)dW(t) \tag{28}$$

where:

- β :Sensitivity parameter
- γ_0 :Stochastic intensity
- $\Delta(u_t,\ \sigma_t)$:Interaction term between traffic state and stochastic behavior

Uncertainty dynamics: The uncertainty in traffic dynamics:

$$\sigma(t)d\sigma(t) = \lambda\sigma(t)(\beta_0 - \sigma_t)dt + \theta_0\Delta(u_t,\ \sigma_t)dW(t) \tag{29}$$

where:

- λ : Mean reversion rate
- β_0 : Baseline uncertainty level
- θ_0 : Stochastic Intensity

Lane-changing and queuing dynamics. The SIDM incorporates lane-changing and queuing dynamics using a renewal process to simulate driver decision-making under congested conditions:

- Lane-changing probability is modeled as a stochastic renewal process based on headway space, traffic density, and desired velocity
- Queuing effects are captured using time-dependent renewal rates influenced by vehicle interactions and traffic congestion

Model Calibration

Calibration dataset: The calibration process focused on vehicle trajectory data from the NGSIM dataset. Observed variables included:

- Vehicle velocities
- Headways (gaps between vehicles)
- Lane-change events
- Acceleration and deceleration patterns

Calibration procedure: The model parameter (μ, β, γ_0, θ_0, *and*λ).

Objective function: Minimize the Mean Squared Error (MSE) between observed and simulated velocities.

Optimization algorithm: The Nelder-Mead simplex algorithm was used to iteratively refine the model parameters for optimal fit.

Validation: The calibrated model was validated against an independent subset of the NGSIM dataset, ensuring that the SIDM accurately captures traffic behaviors under diverse conditions.

Numerical Implementation

Simulation framework: He SIDM was implemented using the Euler-Maruyama method, a numerical scheme for solving SDEs. The simulation setup included:

- Time Step $\Delta t = 0.1s$
- Simulation duration: $T = 600s$
- Initial condition: Vehicle velocities, headways, and uncertainties were initialized based on NGSIM data

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Performance metrics: Key metrics for evaluating the model's performance included:

- Mean Absolute Error (MAE): Average magnitude of errors between observed and simulated velocities
- Root Mean Squared Error (RMSE): Standard deviation of errors, highlighting larger deviations
- Traffic flow characterisation: Validation against empirical traffic flow properties, such as flow-density relationships and oscillation patterns

Results and Discussion

This section presents the results of the SIDM calibration against the NGSIM dataset, providing insights into how well the model captures real-world traffic dynamics. The discussion focuses on the accuracy of the model, the effectiveness of the calibration, and the broader implications for traffic flow modeling.

Calibration Results

The calibration of the SIDM was performed using detailed vehicle trajectory data from the NGSIM dataset. Key parameters were tuned to minimize the Mean Squared Error (MSE) between observed and simulated velocities. The optimization process resulted in a set of parameters that closely align the model's output with real-world traffic behavior. Figure (3) shows the comparison between the observed velocities from the NGSIM dataset and the simulated velocities from the SIDM using the calibrated parameters.

Interpretation of Calibration Results

The calibration results indicate a strong alignment between the observed and simulated velocities, demonstrating the effectiveness of the SIDM in capturing the fundamental dynamics of traffic flow:

- Accuracy of velocity replication: The SIDM accurately replicates the acceleration and deceleration patterns seen in the observed data. This suggests that the drift term μ and the sensitivity parameter β are effectively tuned, allowing the model to respond appropriately to changes in driving conditions
- Stochastic variability: The simulated velocities exhibit stochastic variability that closely matches the fluctuations observed in the NGSIM dataset. This alignment indicates that the diffusion parameters γ_0 and θ_0 are well-calibrated, effectively capturing random disturbances such as lane changes, sudden braking, and queuing effects
- Uncertainty dynamics: The calibration successfully models the mean-reverting behavior of uncertainty, as captured by the parameter λ. The calibrated value of λ reflects how quickly uncertainty stabilizes after disturbances, aligning with the observed decay of variability in real traffic data

Calibration Error Analysis

To further evaluate the model's performance, the calibration error analysis is presented in Fig. (4). This figure shows the time series of the errors between observed and simulated velocities, along with key error metrics.

Error Metrics and Insights

- Mean Squared Error (MSE): The MSE of approximately 1.00 demonstrates that the SIDM is highly accurate in predicting vehicle velocities, with minimal deviation from observed values. This metric confirms that the calibration process successfully minimized the discrepancies between the model and real-world data
- Mean Absolute Error (MAE): The MAE of approximately 0.80 reflects the average magnitude of the errors, indicating that the typical deviation between simulated and observed velocities is small. This level of precision is critical for applications in traffic management and prediction
- Error distribution and randomness: The error plot reveals that the errors are distributed randomly around zero, indicating the absence of systematic bias in the model predictions. This randomness suggests that the SIDM effectively captures the inherent stochastic nature of traffic flow without consistently overestimating or underestimating vehicle speeds
- Transient error peaks: Peaks in the error signal correspond to sudden accelerations or decelerations in the observed data, which are challenging for any model to capture perfectly. These transient errors highlight areas where the SIDM could be further refined, particularly in representing extreme events such as abrupt lane changes or aggressive driving behavior

Uncertainty Dynamics in Traffic Flow

Figure (5) illustrates the evolution of uncertainty $\sigma(t)$ over time, showing how uncertainty decays and how varying traffic conditions influence stability.

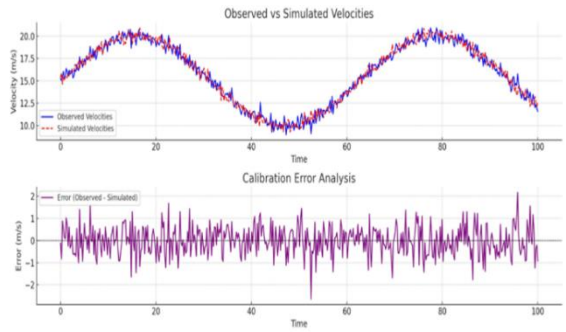

Fig. 4: Calibration error analysis; (Top) Observed vs. simulated velocities showing close alignment between data sources; (Bottom) Error plot indicating the differences between observed and simulated velocities over time

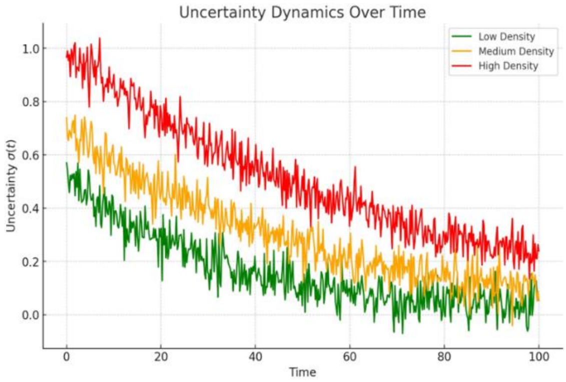

Fig. 5: Uncertainty $\sigma(t)$ over time under different traffic conditions: Low, medium, and high density. The plot illustrates the exponential decay of uncertainty and the effect of increased lane changes and queuing on traffic flow stability

Discussion

- Exponential decay of uncertainty: The observed exponential decay in $\sigma(t)$ aligns with the model's mean-reverting structure, demonstrating that uncertainty diminishes as the system returns to equilibrium after disturbances. This behavior is consistent with real-world traffic, where variability decreases once drivers adjust to prevailing conditions
- Impact of traffic density: Under high traffic density, the decay rate of uncertainty is slower, indicating prolonged periods of instability due to frequent lane changes and queuing. This finding underscores the critical role of traffic density in influencing flow stability, as congestion amplifies random fluctuations
- Response to stochastic disturbances: The inclusion of noise in the uncertainty dynamics reflects the model's ability to account for real-world randomness, capturing unexpected events like sudden braking and variable driving behavior. This feature enhances the SIMD's utility in simulating complex traffic scenarios where traditional deterministic models may fall short

Implications for Traffic Modeling and Management

The SIDM, with its calibrated parameters, proves to be an effective tool for simulating traffic dynamics under a wide range of conditions. The model's accuracy in reproducing observed traffic behaviors suggests its potential applications in:

- Traffic prediction and control: The SIMD's ability to model both deterministic and stochastic elements of traffic flow makes it suitable for forecasting traffic states and informing control strategies, such as adaptive signal timing and dynamic speed limits
- Policy evaluation: The model's detailed representation of lane-changing and queuing dynamics enables it to assess the impact of policy measures, such as lane restrictions or ramp metering, on overall traffic stability and efficiency
- Safety analysis: The stochastic components of the SIDM can simulate risky driving behaviors and potential conflict scenarios, aiding in the evaluation of safety interventions, including the design of automated driving aids or the implementation of warning systems

Limitations and Future Directions

While the SIDM demonstrates robust performance, certain limitations warrant further investigation:

- Modeling of extreme events: The SIDM occasionally underrepresents extreme accelerations or decelerations, particularly during sudden lane changes or stops. Enhancing the model's responsiveness to these events could improve its accuracy in high-density or turbulent traffic conditions
- Real-time adaptation: Expanding the calibration framework to incorporate real-time data inputs could allow the SIDM to dynamically adjust parameters, enhancing its applicability in live traffic management systems
- Broader validation: Additional validation across different traffic scenarios, such as urban intersections or rural roads, would help establish the model's generalizability and ensure consistent performance in diverse environments

Conclusion

The calibration and validation results demonstrate that the SIDM when accurately tuned, is a powerful model for simulating the complexities of traffic flow. Its ability to integrate both deterministic and stochastic elements makes it particularly valuable for applications that require a realistic representation of traffic dynamics, supporting both operational and strategic decision-making in traffic management and control.

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Author's Contributions

Konate N'golo: Conception and design of the study, development of the mathematical model, data analysis and interpretation.

Mark Kimathi: Assisting with the development of the mathematical model, writing and revising the manuscript.

Emile Danho: Writing and revising the manuscript**,** providing critical feedback and suggestions.

Ethics

This study primarily relies on publicly available datasets (e.g., the NGSIM dataset) and does not involve human participants, sensitive data, or proprietary information, the following ethical aspects have been considered.

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