Original Research Paper

Alpha Power Inverted Exponentiated Weibull Distribution with Data Applications

Abdulkareem M. Basheer

Department of Mathematics, Faculty of Administrative Sciences, Albaydha University, Albaydha, Yemen

Article history
Received: 06-04-2024
Revised: 15-05-2024
Accepted: 20-05-2024

Email: abdulkareembasheer@baydaauniv.net

Abstract: A novel extension of the Inverted Exponentiated Weibull (IEW) distribution is achieved through the utilization of the generator Alpha Power (AP) transformation. The resulting extended distribution is denoted as the Alpha Power Inverted Exponentiated Weibull (APIEW) distribution, which encompasses various sub-models. The statistical characteristics of the newly proposed distribution are established, encompassing the hazard rate function, mean residual life, mean inactivity time, quantile function, moments, Rényi entropy, and order statistics. The unknown parameters of the proposed distribution are estimated via the maximum likelihood estimation technique. Subsequently, two sets of application data are employed to demonstrate the adaptability of the model.

Keywords: Inverted Exponentiated Weibull, Alpha Power Transformation, Hazard Rate Function, Moments, Rényi Entropy, Maximum Likelihood Estimation, Simulation

Introduction

The Inverse Weibull (IW) distribution is widely employed in the field of life reliability and testing research. It is considered the reciprocal counterpart of the conventional Weibull distribution, as discussed by (Drapella 1993; Mudholkar and Kollia 1994), This distribution is utilized to characterize the deterioration of mechanical parts within diesel engines, including components like the crankshaft and pistons, as highlighted by Keller *et al*. (1982). The expressions for the Cumulative Distribution Function (CDF) and Probability Density Function (PDF) of the IW distribution can be respectively found in the following equations:

$$F(x) = e^{-vx^{-\eta}}, x \geq 0, v > 0, \eta > 0 \quad (1)$$

And:

$$f(x) = v\eta x^{-(\eta+1)} e^{-vx^{-\eta}}, x \geq 0, v > 0, \eta > 0 \quad (2)$$

where, v is the scale parameter and η is the shape parameter.

Numerous generalizations of the inverse Weibull distribution have been examined in recent years by various researchers. These include the generalized inverse Weibull distribution proposed by De Gusmão *et al*. (2011), the modified inverse Weibull distribution introduced by Khan and King (2012), the beta inverse Weibull model by Hanook *et al*. (2013), the gamma inverse Weibull distribution by Pararai *et al*. (2014), the Kumaraswamy modified inverse Weibull distribution by Aryal and Elbatal (2015), the reflected generalized beta inverse Weibull distribution by Elbatal *et al*. (2016), the Marshall-Olkin extended inverse Weibull distribution by Okasha *et al*. (2017); Okasha *et al*. (2020a-b; 2021; 2022), the Bayesian estimation of Marshall Olkin extended inverse Weibull under progressive type II censoring by Lin *et al*. (2023) and the Generalized modified inverse Weibull distribution by Saboori *et al*. (2020).

The focus is on the Inverted Exponentiated Weibull (IEW) distribution, as introduced by De Gusmão *et al*. (2012); and Lee *et al*. (2017) which is based on the transformation $Z = \frac{1}{X}$, where X follows the Exponentiated Weibull (EW) distribution. The Cumulative Distribution Function (CDF) and Probability Density Function (PDF) of the IEW distribution are provided accordingly:

$$F(x) = 1 - (1 - e^{-vx^{-\eta}})^{\xi}, x \geq 0, v > 0, \eta > 0, \xi > 0 \quad (3)$$

And:

$$f(x) = v\eta\xi x^{-(\eta+1)} e^{-vx^{-\eta}} \left(1 - e^{-vx^{-\eta}}\right)^{\xi-1}, x \geq 0, v > 0, \eta > 0, \xi > 0 \quad (4)$$

where, η and ξ are the shape parameters and v is the scale parameter of IEW distribution. Bayesian parameter

Science Publications

© 2024 Abdulkareem M. Basheer. This open-access article is distributed under a Creative Commons Attribution (CC-BY) 4.0 license.

estimation of the IEW distribution has been previously examined by Lee *et al*. (2017).

In contrast, (Mahdavi and Kundu, 2017) introduced a modification to the underlying Cumulative Distribution Function (CDF) by incorporating an additional parameter to generate a range of distributions. This approach is referred to as the Alpha Power Transformation (APT). If $F(x)$ represents the CDF of any distribution, then the CDF and PDF of the APT can be expressed as:

$$g_{APT}(x)=\begin{cases}\dfrac{\alpha^{F(x)}-1}{\alpha-1}, \alpha>0, \alpha\neq 1\\ F(x), \alpha=1\end{cases} \tag{5}$$

And:

$$g_{APT}(x)=\begin{cases}\dfrac{Log(\alpha)}{\alpha-1} f(x)\alpha^{F(x)}, \alpha>0, \alpha\neq 1\\ f(x), \alpha=1\end{cases} \tag{6}$$

The APT distribution has been extensively studied, with various distributions such as the alpha power Weibull distribution by Nassar *et al*. (2017; 2019), the alpha power Gompertz distribution by Eghwerido *et al*. (2021), the alpha power transformed inverse Lindley distribution by Dey *et al*. (2019), and alpha power inverse Weibull distribution by Basheer *et al*. (2019; 2021; 2022).

The APIEW distribution is introduced in this study as a new modification of the IEW distribution with four parameters. It encompasses a range of lifetime distributions, including the inverse exponential, inverse Rayleigh, IW, alpha power IW, and IEW distributions, as special cases. The APIEW distribution is highlighted due to its inclusion of twelve-lifetime distributions as sub-models and its PDF representation as a mixture of IW distribution, which proves advantageous for deriving its key properties.

Alpha Power Inverted Exponentiated Weibull Distribution

By inserting the CDF of the IEW distribution given by (3) in the CDF of the APT distribution given by (5), we get the CDF of a new distribution denoted as APIEW $(x;\alpha,v,\eta,\xi)$ distribution given by:

$$G_{APT}(x)=\begin{cases}\dfrac{\alpha^{1-\left(1-e^{-vx^{-\eta}}\right)^{\xi}}-1}{\alpha-1}, x\geq 0, \alpha>0, \alpha\neq 1\\ 1-\left(1-e^{-vx^{-\eta}}\right)^{\xi}, \alpha=1\end{cases} \tag{7}$$

where, $v > 0$, $\eta > 0$, $\xi > 0$.

The PDF of APIEW distribution is defined as follows:

$$G_{APT}(x)=\begin{cases}\dfrac{v\eta\xi\log(\alpha)}{\alpha-1}x^{-(\eta+1)}e^{-vx^{-\eta}}\left(1-e^{-vx^{-\eta}}\right)^{\xi-1}\alpha^{1-\left(1-e^{-vx^{-\eta}}\right)^{\xi}}, x\geq 0, \alpha>0, \alpha\neq 1\\ v\eta\xi x^{-(\eta+1)}e^{-vx^{-\eta}}\left(1-e^{-vx^{-\eta}}\right)^{\xi-1}, \alpha=1\end{cases} \tag{8}$$

where, $v > 0$, $\eta > 0$, $\xi > 0$.

Through the application of the generalized binomial expansion and the power series, a valuable linear portrayal of the Probability Density Function (PDF) is derived. (if $\alpha > 0$, $\alpha \neq 1$) as:

$$g_{APIEW}(x)=\sum_{m=0}^{\infty}W_m v\eta(m+1)x^{-(\eta+1)}e^{-(m+1)vx^{-\eta}} \tag{9}$$

where:

$$W_m=\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j+m}\binom{k}{j}\frac{\xi(\xi j+\xi-1)!(\log(\alpha))^{k+1}}{(\xi j+\xi-m-1)!(m+1)!k!(\alpha-1)}$$

Several sub-models of the APIEW distribution are enumerated in the Table (1).

Figure (1) provides a graphical illustration of the PDF corresponding to various parameter values.

Reliability Analysis

The reliability function of APIEW distribution is defined as follows:

$$R(t)=\begin{cases}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{-\left(1-e^{-vt^{-\eta}}\right)^{\xi}}\right), t\geq 0, \alpha>0, \alpha\neq 1\\ \left(1-e^{-vt^{-\eta}}\right)^{\xi}, \alpha=1\end{cases} \tag{10}$$

Hazard Rate Function

The HR function of APIEW distribution is defined as follows:

$$h(t)=\frac{g(t)}{R(t)}=\begin{cases}\log(\alpha)v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}\dfrac{\left(1-e^{-vt^{-\eta}}\right)^{\xi-1}}{-1+\alpha\left(1-e^{-vt^{-\eta}}\right)^{\xi}}, t\geq 0, \alpha>0, \alpha\neq 1\\ \dfrac{v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}}{1-e^{-vt^{-\eta}}}, \alpha=1\end{cases} \tag{11}$$

The graphical representations of the HRF for various parameter values are illustrated in Fig. (2).

Reversed Hazard Rate Function

The reversed hazard rate (RHR) function of APIEW distribution is defined as follows:

$$r(t)=\frac{g(t)}{G(t)}=\begin{cases}\log(\alpha)v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}\dfrac{\left(1-e^{-vt^{-\eta}}\right)^{\xi-1}}{1-\alpha^{-1+\left(1-e^{-vt^{-\eta}}\right)^{\xi}}}, t\geq 0, \alpha>0, \alpha\neq 1\\ \dfrac{v\eta\xi t^{-(\eta+1)}e^{-vt^{-\eta}}\left(1-e^{-vt^{-\eta}}\right)^{\xi-1}}{1-\left(1-e^{-vt^{-\eta}}\right)^{\xi}}, \alpha=1\end{cases} \tag{12}$$

DOI: 10.3844/jmssp.2024.63.73

Table 1: Sub-models of the APIEW (α, ν, η, ξ) distribution

Models	Parameters			
Inverted Exponentiated Weibull (IEW)	1	ν	η	ξ
Inverted Exponentiated Fréchet (IEF)	1	1	η	ξ
Inverted Exponentiated Rayleigh (IER)	1	ν	2	ξ
Inverted Exponentiated Exponential (IEE)	1	ν	1	ξ
Alpha Power Inverse Weibull (APIW)	α	ν	η	1
Alpha Power Fréchet (APF)	α	1	η	1
Alpha Power Inverse Rayleigh (APIR)	α	ν	2	1
Alpha Power Inverse Exponential (APIE)	α	ν	1	1
Inverse Weibull (IW)	1	ν	η	1
Fréchet (F)	1	1	η	1
Inverse Rayleigh (IR)	1	ν	2	1
Inverse Exponential (IE)	1	ν	1	1

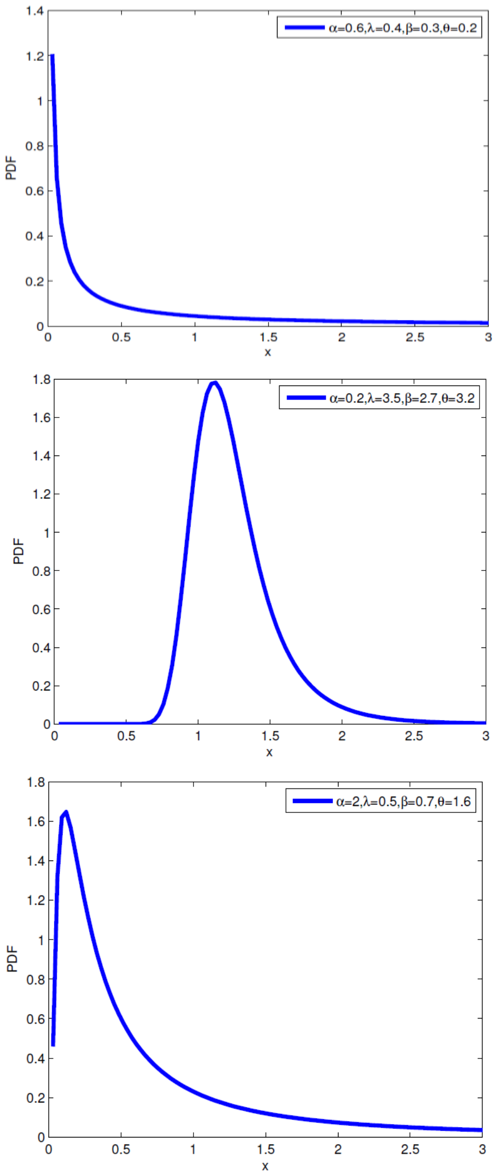


Fig. 1: Plot of the PDF of the APIEW distribution for some values of parameters

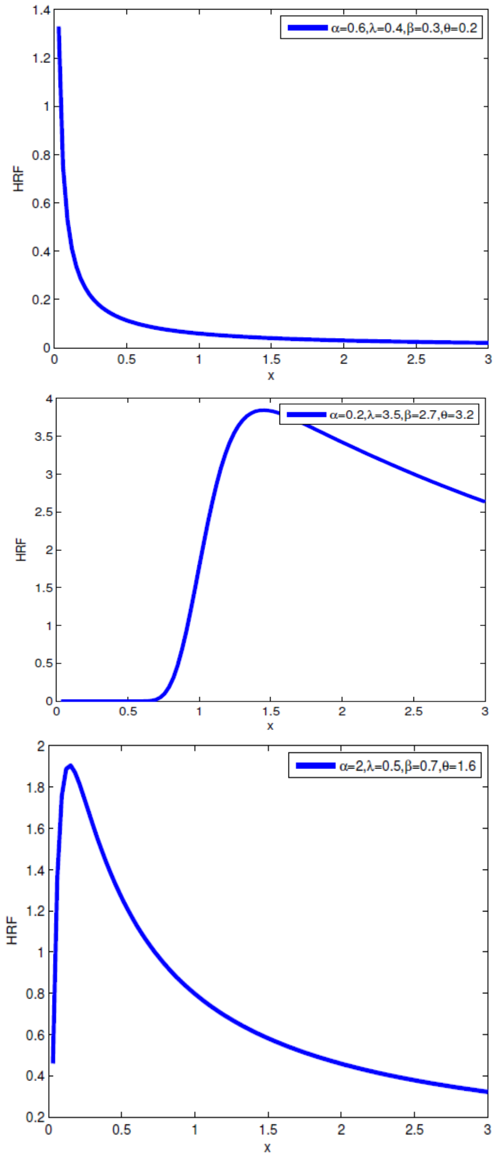


Fig. 2: Plot of the HRF of the APIEW distribution for some values of parameters

Table 2: Some reliability of APIEW for selected values of $\lambda = 1.3$ and $\beta = 5$ at $t = 0.8$

α	ξ	HRF	MRL	RHR	MIT	SMIT
0.3	1.5	1.21717	0.24858	24.2596	0.032402	0.050077
	2.3	1.85296	0.18546	23.8596	0.032640	0.050439
0.8	1.5	0.79804	0.30161	24.5992	0.032202	0.049775
	2.3	1.22173	0.22266	24.3706	0.032337	0.049977
1.4	1.5	0.60654	0.33461	24.7944	0.032088	0.049602
	2.3	0.93246	0.24550	24.6654	0.032162	0.049714
2.6	1.5	0.43518	0.37224	25.0115	0.031963	0.049412
	2.3	0.67272	0.27133	24.9943	0.031971	0.049424

Mean Residual Life

The MRL function is defined as follows:

$$\mu(t) = \frac{1}{R(t)}\int_t^\infty xg(x)\,dx - t, t \geq 0$$

Proposition 3.1. The MRL function for a lifetime random variable X following the APIEW distribution can be expressed as:

$$\mu(t) = \frac{1}{R(t)} = \sum_{m=0}^{\infty} W_m v^{\frac{1}{\eta}} (m+1)^{\frac{1}{\eta}} \gamma\left(1-\frac{1}{\eta}, v(m+1)t^{-\eta}\right) - t, \eta > 1 \quad (13)$$

Proof.

By employing the definition of MRL and using (9), we get:

$$\mu(t) = \frac{1}{R(t)}\int_t^\infty xg(x)dx - t$$
$$= \frac{1}{R(t)}\sum_{m=0}^{\infty} w_m \int_t^\infty x.\,v\eta\,(m+1)x^{-(\eta+1)}e^{-v(m+1)x^{-\eta}}dx - t$$

Put $z = v(m+1)x^{-\eta}$ thus

$$\mu(t) = \frac{1}{R(t)} = \sum_{m=0}^{\infty} W_m v^{\frac{1}{\eta}} (m+1)^{\frac{1}{\eta}} \gamma\left(1-\frac{1}{\eta}, v(m+1)t^{-\eta}\right) - t$$

where, $\gamma(s,t) = \int_0^t x^{s-1}e^{-x}\,dx, s > 0$

Mean Inactivity Time

The MIT function is defined as follows:

$$m(t) = t - \frac{1}{G(t)}\int_0^t xg(x)\,dx, t \geq 0$$

Proposition 3.2. The MIT function of a lifetime random variable X with APIEW is given by:

$$m(t) = t - \frac{1}{G(t)}\sum_{m=0}^{\infty} W_m v^{\frac{1}{\eta}} (m+1)^{\frac{1}{\eta}} \Gamma\left(1-\frac{1}{\eta}, v(m+1)t^{-\eta}\right), \eta > 1 \quad (14)$$

Proof.

By employing the proof of the MRL and the relation $\Gamma(c,t) = \int_t^\infty x^{c-1}e^{-x}dx, c > 0$, we get the above result.

Strong Mean Inactivity Time

The Strong Mean Inactivity Time (SMIT) represents a novel reliability metric introduced by the work by Kayid and Izadkhah (2014). The definition of the SMIT:

$$M(t) = \frac{1}{G(t)}\int_0^t 2xG(x)\,dx = t^2 - \frac{1}{G(t)}\int_0^t x^2\,g(x)dx, t \geq 0$$

The SMIT function of APIEW distribution is:

$$M(t) = t^2 - \frac{1}{G(t)}\sum_{m=0}^{\infty} W_m v^{\frac{2}{\eta}} (m+1)^{\frac{2}{\eta}} \Gamma\left(1-\frac{2}{\eta}, v(m+1)t^{-\eta}\right), \eta > 2 \quad (15)$$

Table (2) Provides the numerical data pertaining to HRF, MRL, RHR, and MIT (SMIT) corresponding to the specific set of selected parameters $v = 1.3$, $\eta = 5$, and $t = 0.8$ for various parameter values. α and ξ. Also, from Table (2) we see the:

- The decrease in HRF is observed with the increase in the MRL
- The increases in RHR are observed with the decreases in the MIT(SMIT)

Statistical Properties

In this section, the statistical properties of the APIEW distribution are examined, focusing on the quantile function, moments, moment generating function, entropy, and order statistics.

Quantile Function

The solution for the quantile function of a distribution is obtained by solving the equation:

$$G(x_p) = p, 0 < p < 1 \quad (16)$$

The quantile function of the APIEW distribution can be expressed as follows proposition.

Proposition 4.1. If a random variable X follows an APIEW (α,v,η,ξ) distribution, then the quantile function of X can be determined by:

$$x_p = G^{-1}(p) = \left[\frac{-1}{v}\log\left(1-\left(1-\frac{\log(p\alpha - p + 1)}{\log(\alpha)}\right)^{\frac{1}{\xi}}\right)\right]^{-1/\eta} \quad (17)$$

Proof.

By considering the function $h-1-e^{-vx^{-\eta}}$, the CDF of the APIEW distribution $G(x)=\frac{\alpha^{1-h^{\xi}}-1}{\alpha-1}$.

The p^{th} quantile function is derived by solving $G(x) = p$ and the obtained result is $h-1-e^{-vx^{-\eta}}$ by solving for x we get:

$$x_p=G^{-1}(p)=\left[\frac{-1}{v}\log\left(1-\left(1-\frac{\log(p\alpha-p+1)}{\log(\alpha)}\right)^{\frac{1}{\xi}}\right)\right]^{-1/\eta}$$

Statistical measures for the APIEW distribution can be calculated based on Eq. (17), such as obtaining the 1st quartile for $p = 0.25$, the median for $p = 0.5$, and the 3rd quartile for $p = 0.75$. In order to generate samples for the APIEW distribution, Eq. (17) can be utilized.

Moments

The r^{th} moments of the APIEW distribution are given by the following proposition.

Proposition 4.2. If a random variable X follows an APIEW (α,v,η,ξ) distribution, then the r^{th} moments of X can be determined by:

$$\mu_r'=E\left(X^r\right)=\sum_{m=0}^{\infty}W_m v^{\frac{r}{\eta}}(m+1)^{\frac{r}{\eta}}\Gamma\left(1-\frac{r}{\eta}\right),\eta>r \tag{18}$$

Proof.

From the definition of moments and utilizing Eq. (7), it can be derived:

$$\mu_r'=E\left(X^r\right)=\sum_{m=0}^{\infty}W_m v\eta(m+1)\int_0^{\infty}x^{r-(\eta+1)}e^{-v(m+1)x^{-\eta}}dx$$

Put $Z=v(m+1)x^{-\eta}$ Thus

$$\mu_r'=E\left(X^r\right)=\sum_{m=0}^{\infty}W_m v^{\frac{r}{\eta}}(m+1)^{\frac{r}{\eta}}\Gamma\left(1-\frac{r}{\eta}\right),\eta>r$$

where, $\Gamma(.)$ denotes the gamma function. Specifically, the initial two moments can be calculated as:

$$\mu_1'=E(X)=\sum_{m=0}^{\infty}W_m v^{\frac{1}{\eta}}(m+1)^{\frac{1}{\eta}}\Gamma\left(1-\frac{1}{\eta}\right),\eta>1$$

And:

$$\mu_2'=E\left(X^2\right)=\sum_{m=0}^{\infty}W_m v^{\frac{2}{\eta}}(m+1)^{\frac{2}{\eta}}\Gamma\left(1-\frac{2}{\eta}\right),\eta>2$$

The subsequent formulas can also be utilized to calculate the mean, variance, skewness, and kurtosis:

$$Mean=\mu_1'=\mu,$$

$$uariance=\mu_2'-\mu^2,$$

$$skewness=\frac{\mu_3'-3\mu_2'\mu+2\mu^3}{\left(\mu_2'-\mu^2\right)^{\frac{3}{2}}}\,and$$

$$kurtosis=\frac{\mu_4'-4\mu_2'\mu+6\mu_2'\mu^2-3\mu^4}{\left(\mu_2'-\mu^2\right)^2}$$

Table (3) Gives the median, skewness, kurtosis and moments of APIEW distribution for specific parameters $v = 1.3$ and $\eta = 5$. along with various values of the parameters α and ξ.

Moment Generating Function

The next proposition provides the moment-generating function (MGF) of the APIEW distribution.

Proposition 4.3. If a random variable X follows an APIEW (α,v,η,ξ) distribution, then the MGF of X can be determined by:

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^r}{r!}W_m v^{\frac{r}{\eta}}(m+1)^{\frac{r}{\eta}}\Gamma\left(1-\frac{r}{\eta}\right),\eta>r \tag{19}$$

Proof.

We can express:

$$M_X(t)=\int_0^{\infty}e^{tx}\,g(x)dx$$

Upon utilizing the Taylor's series expansion of the function e^{tx}, the expression simplifies to:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty}x^r\,g(x)dx$$

By applying the same method used for proving moments, the result presented above is obtained.

Rényi Entropy

Rényi entropy of order δ is defined as:

$$H_\delta=\frac{1}{1-\delta}\log(\int_{-\infty}^{\infty}(g(x))^\delta dx),\delta\geq 0,\delta\neq 1$$

Let $X\sim APIEW(x;\alpha,v,\eta,\xi)$ then:

$$H_\delta=\frac{1}{1-\delta}\log\left(\int_0^{\infty}\left(\frac{\log(\alpha)}{\alpha-1}v\eta\xi x^{-(\eta+1)}e^{-vx^{-\eta}}\times\left(1-e^{-vx^{-\eta}}\right)^{\xi-1}\alpha^{1-\left(1-e^{-vx^{-\eta}}\right)^{\xi}}\right)^{\delta}dx\right) \tag{20}$$

$$=\frac{1}{1-\delta}\log\left[\sum_{m1=0}^{\infty}W_{m1}(\delta+m1)^{-\frac{\delta\eta+\delta-1}{\eta}}\Gamma\left(\frac{\delta\eta+\delta-1}{\eta}\right)\right]$$

where:

$$W_{m1}=\sum_{k1=0}^{\infty}\sum_{j1=0}^{k1}(-1)^{j1+m1}\binom{k1}{j1}\binom{j1\xi+\delta(\xi-1)}{m1}\frac{(\log(\alpha))^{\delta+k1}}{(\alpha-1)^{\delta}k1!}v^{\frac{\delta-1}{\eta}}\delta^{k1}\eta^{\delta-1}\xi^{\delta}$$

Table 3: Median and moments of APIEW for selected values of v = 1.3 and n = 5

α	ξ	Median	Mean	Variance	Skewness	Kurtosis
0.3	1.5	0.99044	1.03516	0.04331	2.25544	15.1943
	2.3	0.94313	0.96974	0.02181	1.50423	7.89711
0.8	1.5	1.04191	1.09113	0.05562	2.06334	13.3022
	2.3	0.98298	1.01049	0.02657	1.34908	7.00147
1.4	1.5	1.07581	1.12584	0.06236	1.97078	12.5658
	2.3	1.00861	1.03539	0.02894	1.26953	6.65369
2.6	1.5	1.11538	1.16533	0.06911	1.89135	12.0279
	2.3	1.03788	1.06339	0.03105	1.19840	6.41016

Table (4) presents the Rényi entropy values corresponding to the chosen parameters (v = 1.3 and η = 2.1) and various values of α, ξ, and δ. Also, from Table (4) we see that the Rényi entropy increases when the α increases (ξ decreases).

Order Statistics

The order statistics of a random sample $X_1,..., X_n$ refer to the ordered sample values. They are typically denoted as $X_{1:n},..., X_{n:n}$. The PDF of the i^{th} order statistic $X_{i:n}$ can be expressed as:

$$g_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} g(x)[G(x)]^{i-1}[1-G(x)]^{n-i}, i = 1,\ldots,n \quad (21)$$

Hence the PDF of the i^{th} order statistics $X_{i:n}$ of APIEW distribution can be obtained by substituting from (7) and (8) into (21), we get the i^{th} order statistics of APIEW density function as follows:

$$g_{i:n}(x) = \frac{n! v\eta\xi \log(\alpha)}{(i-1)!(n-i)!\alpha - 1} x^{-(\eta+1)}\Psi(1-\Psi)^{\xi-1}\alpha^{1-(1-\Psi)^{\xi}} \times \left(\frac{\alpha^{1-(1-\Psi)\xi} - 1}{\alpha - 1}\right)^{i-1}\left(\frac{\alpha}{\alpha-1}\left(1-\alpha^{-(1-\Psi)^{\xi}}\right)\right)^{n-i} \quad (22)$$

where $\Psi = e^{-vx-\eta}$.

The CDF of the i^{th} order statistics $X_{i:n}$ can be expressed as:

$$G_{i:n}(x) = \sum_{s=i}^{n}\binom{n}{s}\left[G(x)\right]^{s}\left[1-G(x)\right]^{n-s} \quad (23)$$

Hence the CDF of the i^{th} order statistics $X_{i:n}$ of APIEW distribution can be obtained by substituting from (7) into (23).

Materials and Methods

Maximum Likelihood Estimation Method

This subsection discusses the Maximum Likelihood Estimation (MLE) for the parameters $\Theta = (\alpha,v,\eta,\xi)$ of the APIEW distribution. Consider a complete random sample of size n from the APIEW distribution, denoted as $x_1,x_2,...,x_n$. The likelihood function can be expressed as:

$$\ell(x_1,\ldots,x_n|\Theta) = \prod_{i=1}^{n} g(x_i) = \prod_{i=1}^{n}\frac{\log(\alpha)}{\alpha-1} v\eta\xi x_i^{-(\eta+1)} e^{-vx_i^{-\eta}}\left(1-e^{-vx_i^{-\eta}}\right)^{\xi-1}\alpha^{1-\left(1-e^{-vx_i^{-\eta}}\right)^{\xi}}$$

Then, the logarithm of the likelihood function is given by:

$$L(x_1,\ldots,x_n|\Theta) = n\log(\log(\alpha)) - n\log(\alpha-1) + n\log(v\eta\xi) - (\eta+1)\sum_{i=1}^{n}\log(x_i) - v\sum_{i=1}^{n}x_i^{-\eta} + (\xi-1)\sum_{i=1}^{n}\log\left(1-e^{-vx_i^{-\eta}}\right) + \log(\alpha)\sum_{i=1}^{n}\left(1-\left(e^{-vx_i^{-\eta}}\right)^{\xi}\right) \quad (24)$$

Upon deriving the first partial derivatives of the log-likelihood function with respect to the parameters in Θ, we obtain a set of equations:

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha\log(\alpha)} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-(1-\Psi_i)^{\xi}\right) = 0 \quad (25)$$

$$\frac{\partial L}{\partial v} = \frac{n}{v} - \sum_{i=1}^{n}x_i^{-\eta} + (\xi-1)\sum_{i=1}^{n}\frac{\Psi_i x_i^{-\eta}}{1-\Psi_i} - \xi\log(\alpha)\sum_{i=1}^{n}x_i^{-\eta}\Psi_i(1-\Psi_i)^{\xi-1} = 0$$
(26)

$$\frac{\partial L}{\partial \eta} = \frac{n}{\eta} - \sum_{i=1}^{n}\log(x_i) + v\sum_{i=1}^{n}x_i^{-\eta}\log(x_i) - v(\xi-1)\sum_{i=1}^{n}\frac{\Psi_i x_i^{-\eta}\log(x_i)}{1-\Psi_i} + v\xi\log(\alpha)\sum_{i=1}^{n}\Psi_i(1-\Psi_i)^{\xi-1}x_i^{-\eta}\log(x_i) = 0 \quad (27)$$

and

$$\frac{\partial L}{\partial \eta} = \frac{n}{\xi} - \sum_{i=1}^{n}\log(1-\Psi_i) - \log(\alpha)\sum_{i=1}^{n}(1-\Psi_i)^{\xi}\log(1-\Psi_i) = 0 \quad (28)$$

where $\Psi_i = e{-}v_x{-}i\ \eta$.

Due to the complexity of these equations, explicit solutions are not feasible. Therefore, numerical methods become necessary for the estimation of the MLEs of the parameters $\Theta = (\alpha,v,\eta,\xi)$.

Table 4: Values of Rényi entropy of APIEW distribution

δ	α	ξ	R´enyi entropy
1.2	0.3	0.2	2.92597
		1.5	0.42441
	0.8	0.2	3.56940
		1.5	0.64175
	1.4	0.2	3.96504
		1.5	0.75677
2.3	0.3	0.2	1.99853
		1.5	0.16046
	0.8	0.2	2.46871
		1.5	0.38243
	1.4	0.2	2.77052
		1.5	0.50334

Simulation

In this subsection, we examined the behavior of MLEs derived from unspecified parameters. The simulation was carried out using the Mathematica program and the following is the technique that was followed for it:

1. Two different sets of initial parameter values are considered set(A): $\alpha = 0.6$, $\nu = 0.8$, $\eta = 0.5$ and $\xi = 0.3$ and set (B): $\alpha = 0.3$, $\nu = 0.5$, $\eta = 0.7$ and $\xi = 0.6$
2. 1000 random samples from different sample sizes $n = 50, 100, 150, 200, 300$ are generated using the (17)
3. The calculated Mean Squared Error (MSE) as well as the Bias are then presented when the result has been obtained

The MSE and bias of respective estimators are given by:

$$MES(\hat{\theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta} - \hat{\theta})^2, Bias(\hat{\theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\theta} - \hat{\theta})$$

where $\Theta = (\hat{\alpha}, \hat{\nu}, \hat{\eta}, \hat{\xi})$.

Table (5) Illustrates the Mean Squared Error (MSE) and the bias value of the parameters. Moreover, it is evident from Table (5) that the MSE (Bias) diminishes with the increment in sample size.

Real Data

In this section, we conduct an analysis of empirical data to demonstrate the efficacy of the APIEW as a viable model for lifetime estimation, in comparison to established distributions such as Alpha Power Inverse Weibull (APIW), Inverted Exponentiated Weibull (IEW) and Inverse Weibull (IW) distributions.

The first data set pertains to the mortality trends attributed to the COVID-19 outbreak in the United Kingdom over a span of 76 days, spanning from 15th April to 30th June 2020. This dataset was initially scrutinized by Mubarak and Al-Metwally, (2021). The second dataset delineates the durations of waiting (in minutes) prior to receiving customer assistance in a financial institution. The second dataset has been initially scrutinized by Ghitany *et al.* (2008). The first data and second data are displayed in Table (6).

The MLEs of the APIEW distribution as well as several other competing distributions are showcased in Tables (7-8) for the first and second datasets, respectively. Furthermore, Tables (7) and 8 also present various goodness-of-fit metrics such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Kolmogorov-Smirnov (K-S) statistic along with their corresponding p-values for both sets of data. Analysis of these tables reveals that the APIEW distribution outperformed all other competitive distributions, establishing itself as the most suitable model for fitting the provided datasets. Figures (3-4) exhibit the fitted PDFs, CDFs, RFs, and PP plots for the APIEW distribution with respect to the first and second datasets. These visual representations illustrate the capacity of the APIEW distribution to closely approximate the given datasets.

Table 5: MLE of parameters α, λ, β and θ

APIEW ($\alpha, \lambda, \beta, \theta$)	n	MSE($\hat{\alpha}$) Bias($\hat{\alpha}$)	MSE($\hat{\nu}$) Bias($\hat{\nu}$)	MSE($\hat{\eta}$) Bias($\hat{\eta}$)	MSE($\hat{\xi}$) Bias($\hat{\xi}$)
	50	0.637984	0.641329	0.100141	0.560831
		0.384949	0.101142	0.113329	0.118349
	100	0.586985	0.306908	0.058644	0.233698
		0.295446	-0.095049	0.093508	0.027588
APIEW (0.6,0 8,0.5,0.3)	150	0.506365	0.235828	0.037153	0.132643
		0.258647-	0.079209	0.072439	-0.018376
	150	0.506365	0.235828	0.037153	0.132643
		0.258647-	0.079209	0.072439	-0.018376
	200	0.487527	0.155893	0.035711	0.065784
		0.236805	-0.050396	0.068773	-0.006879
	300	0.389554	0.113611	0.021887	0.019242
		0.219046	-0.031330	0.045049	-0.003664
	50	0.960334	1.276341	0.209416	0.704751
		0.534367	0.232847	0.172251	0.091041
	100	0.810275	0.890308	0.129319	0.672735
		0.497137	0.112601	0.141588	0.075258
APIEW (0.3,0.5, 0.7,0.6)	150	0.703661	0.660812	0.102225	0.652296
		0.438629	0.085103	0.109805	0.058749
	200	0.650309	0.534065	0.078401	0.589692
		0.395532	0.061709	0.091561	0.040322
	300	0.203455	0.117308	0.026731	0.005775
		0.075293	-0.058479	0.058635	-0.009791

Table 6: The first data and second data

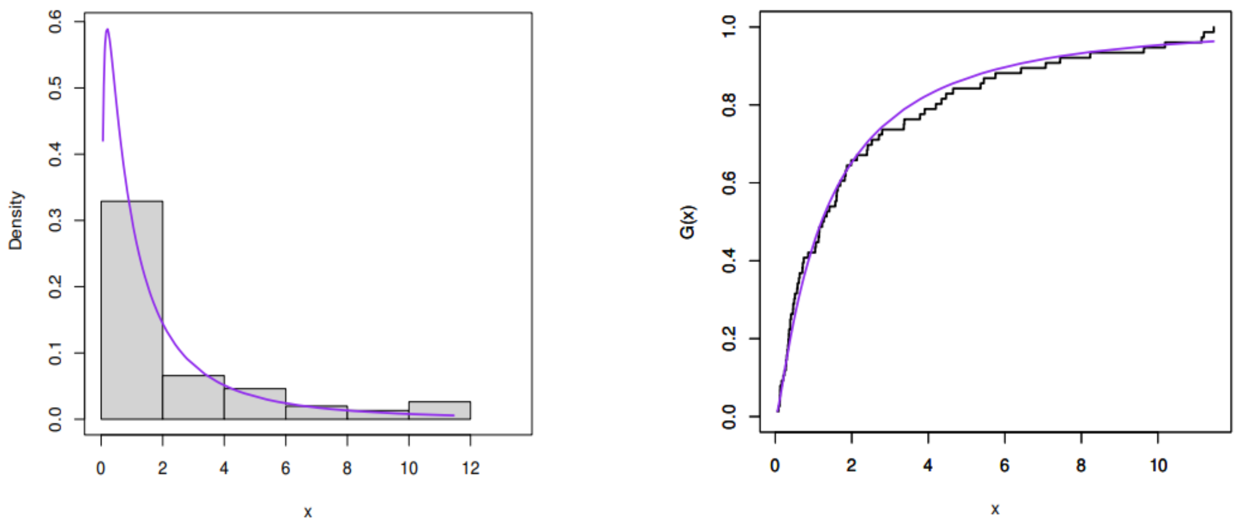
First data	0.0587	0.0863	0.1165	0.1247	0.1277	0.1303	0.1652	0.2079	0.2395
	0.2751	0.2845	0.2992	0.3188	0.3317	0.3446	0.3553	0.3622	0.3926
	0.3926	0.4110	0.4633	0.4690	0.4954	0.5139	0.5696	0.5837	0.6197
	0.6365	0.7096	0.7193	0.7444	0.8590	1.0438	1.0602	1.1305	1.1468
	1.1533	1.2260	1.2707	1.3423	1.4149	1.5709	1.6017	1.6083	1.6324
	1.6998	1.8164	1.8392	1.8721	1.9844	2.1360	2.3987	2.4153	2.5220
	2.7087	2.7946	3.3609	3.3715	3.7840	3.9042	4.1969	4.3451	4.4627
	4.6477	5.3664	5.4500	5.7522	6.4241	7.0657	7.4456	8.2307	9.6315
	10.187	11.1429	11.2019	11.4584					
Second data	0.80	0.80	1.30	1.50	1.80	1.90	1.90	2.10	2.60
	2.70	2.90	3.10	3.20	3.30	3.50	3.60	4.00	4.10
	4.20	4.20	4.30	4.30	4.40	4.40	4.60	4.70	4.70
	4.80	4.90	4.90	5.00	5.30	5.50	5.70	6.10	6.20
	6.20	6.20	6.30	6.70	6.70	6.90	7.10	7.10	7.10
	7.10	7.40	7.60	7.70	8.00	8.20	8.60	8.60	8.60
	8.80	8.80	8.90	8.90	9.50	9.60	9.70	9.80	10.70
	10.90	11.00	11.00	11.10	11.20	11.20	11.50	11.90	12.40
	12.50	12.90	13.00	13.10	13.30	13.60	13.70	13.90	14.10
	15.40	15.40	17.30	17.30	18.10	18.20	18.40	18.90	19.00
	19.90	20.60	21.30	21.40	21.90	23.00	27.00	31.60	33.10
	38.50								

Table 7: MLEs and different statistics of APIEW for first data

	Estimates				Statistics			
Distributions	α	ν	η	ξ	AIC	BIC	K-S	P-Value
APIEW	0.9186	3.805	0.242	25.18	287.538	296.861	0.0563	0.9693
APIW	17.523	0.2617	0.9450	—	292.04	299.03	0.0796	0.7208
IEW	—	1.621	0.46821	3.231	289.19	296.18	0.1009	0.4210
IW	—	0.6701	0.7896	—	294.34	299	0.1021	0.4059

Table 8: MLEs and different statistics of APIEW for second data

	Estimates				Statistics			
Distributions	α	ν	η	ξ	AIC	BIC	K-S	P-Value
APIEW	32.317	4.474	1.020	2.1	652.48	662.90	0.0549	0.9231
APIW	98.324	3.076	1.487	—	658.61	666.43	0.0876	0.4261
IEW	—	7.832	1.1	1.49	668.52	676.34	0.1048	0.2218
IW	—	6.533	1.163	—	672.76	677.97	0.1166	0.1314



DOI: 10.3844/jmssp.2024.63.73

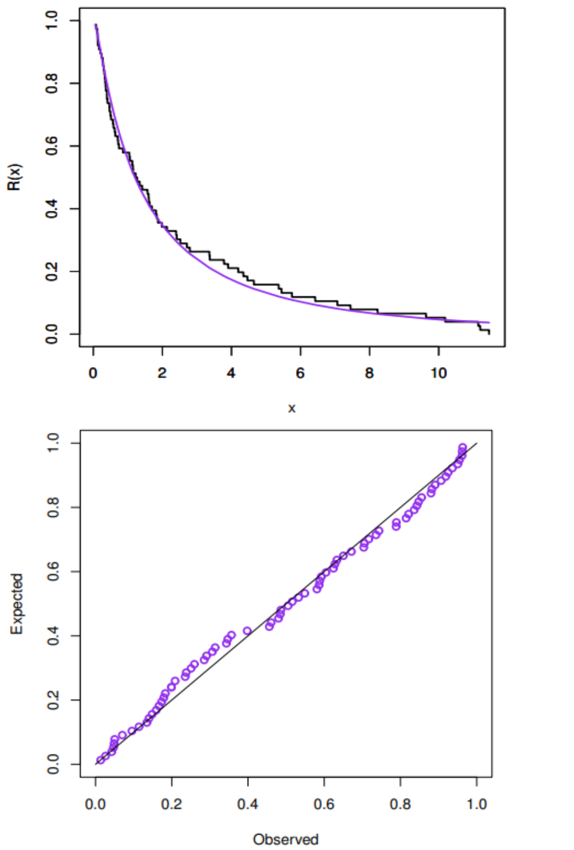
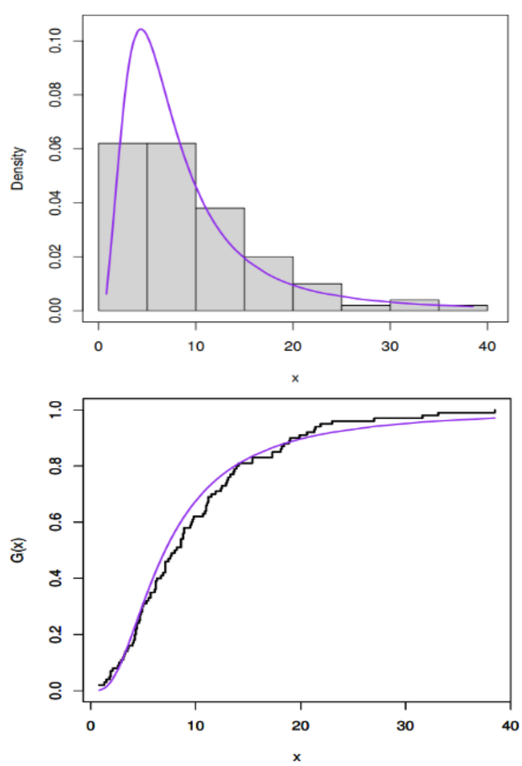


Fig. 3: Plots of the fitted functions for the APIEW distribution and PP plot for the first data



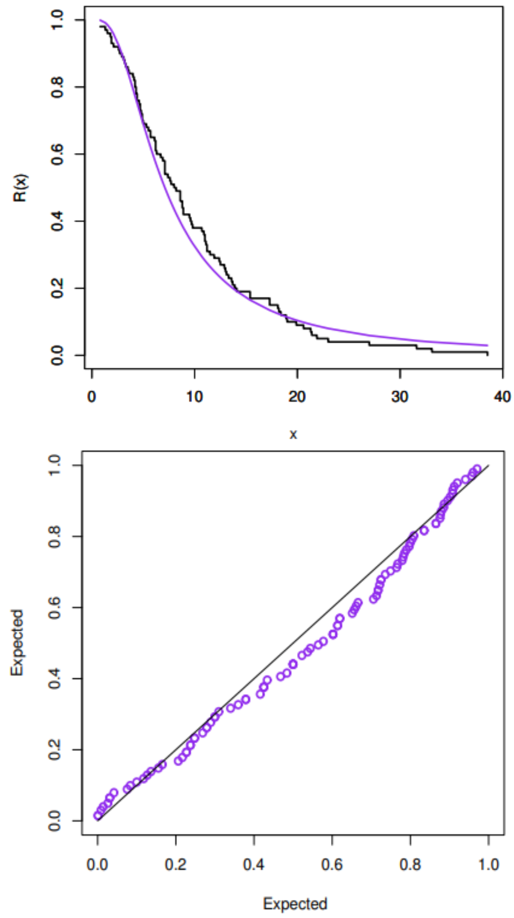


Fig. 4: Plots of the fitted functions for the APIEW distribution and PP plot for the second data

Results and Discussion

Table (5) shows the numerical results of the simulation applied to our model which had four parameters using the maximum likelihood estimation method where the results showed the small MSE and bias of these parameters. Also, Tables (7-8) applied to real data by our model show the suitability of these data to our proposed model and Figs (3-4) support this. Thus, our model appears to be a better fit for this data set than many existing models.

Conclusion

Contemporary research has made a significant contribution by introducing a novel extended distribution utilizing the Alpha Power (AP) transformation on the Inverted Exponentiated Weibull (IEW) distribution. Referred to as the APIEW distribution, this new distribution serves as a generalization of the IEW distribution. Various statistical characteristics of the APIEW distribution have been obtained and deliberated upon, such as the hazard rate function, mean residual life, mean inactivity time, quantile function, moments, Rényi entropy, and order statistics. Moreover, the maximum likelihood estimation method has been suggested for estimating the parameters of the APIEW distribution, with the outcomes of a simulation study endorsing the effectiveness of the MLE method in parameter estimation. The efficacy of the model has been exhibited through the application of two real datasets, showcasing its practical utility. The proposed distribution emerges as a more suitable model for fitting such datasets compared to numerous existing models and recently developed distributions. Subsequent research endeavors may focus on exploring the estimation challenges of the proposed model under progressive type II censoring. Additionally, a comparison could be made between the traditional parameter estimation techniques, such as the maximum product of spacing and least squares methods, utilizing the squared error loss and LINEX loss functions.

Acknowledgment

Thank you to the publisher for their support in the publication of this study and thank the editor, associate editor and anonymous referees for their suggestions and comments that led to a significant improvement of this study.

Funding Information

The authors have not been provided with any financial assistance or funding to report.

Ethics

This manuscript presents original content comprising unpublished material. The primary author attests that all co-authors have reviewed and endorsed the document, and no ethical considerations have been raised.

References

Aryal, G., & Elbatal, I. (2015). Kumaraswamy Modified Inverse Weibull Distribution. *Applied Mathematics and Information Sciences*, *9*(2), 651–660. https://doi.org/10.12785/amis/090213

Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. *Journal of Taibah University for Science*, *13*(1), 423–432. https://doi.org/10.1080/16583655.2019.1588488

Basheer, A. M. (2022). Marshall–Olkin Alpha Power Inverse Exponential Distribution: Properties and Applications. *Annals of Data Science*, *9*(2), 301–313. https://doi.org/10.1007/s40745-019-00229-0

Basheer, A. M., Almetwally, E. M., & Okasha, H. M. (2021). Marshall-Olkin Alpha Power Inverse Weibull Distribution: Non-Bayesian and Bayesian Estimations. *Journal of Statistics Applications & Probability*, *10*(2), 327–345. https://doi.org/10.18576/jsap/100205

De Gusmão, F. R. S., Ortega, E. M. M., & Cordeiro, G. M. (2011). The Generalized Inverse Weibull Distribution. *Statistical Papers*, *52*(3), 591–619. https://doi.org/10.1007/s00362-009-0271-3

De Gusmão, F. R. S., Ortega, E. M. M., & Cordeiro, G. M. (2012). Reply to the “Letter to the Editor” of M. C. Jones. *Statistical Papers*, *53*(2), 253–254. https://doi.org/10.1007/s00362-012-0441-6

Dey, S., Nassar, M., & Kumar, D. (2019). Alpha Power Transformed Inverse Lindley Distribution: A Distribution with an Upside-Down Bathtub-Shaped Hazard Function. *Journal of Computational and Applied Mathematics*, *348*, 130–145. https://doi.org/10.1016/j.cam.2018.03.037

Drapella, A. (1993). The Complementary Weibull Distribution: Unknown or Just Forgotten? *Quality and Reliability Engineering International*, *9*(4), 383–385. https://doi.org/10.1002/qre.4680090426

Eghwerido, J. T., Nzei, L. C., & Agu, F. I. (2021). The Alpha Power Gompertz Distribution: Characterization, Properties and Applications. *Sankhya A*, *83*(1), 449–475. https://doi.org/10.1007/s13171-020-00198-0

Elbatal, I., Condino, F., & Domma, F. (2016). Reflected Generalized Beta Inverse Weibull Distribution: definition and properties. *Sankhya B*, *78*(2), 316–340. https://doi.org/10.1007/s13571-015-0114-2

Ghitany, M. E., Atieh, B., & Nadarajah, S. (2008). Lindley Distribution and Its Application. *Mathematics and Computers in Simulation*, *78*(4), 493–506. https://doi.org/10.1016/j.matcom.2007.06.007

Hanook, S., Shahbaz, M. Q., Mohsin, M., & Golam Kibria, B. M. (2013). A Note on Beta Inverse-Weibull Distribution. *Communications in Statistics - Theory and Methods*, *42*(2), 320–335. https://doi.org/10.1080/03610926.2011.581788

Kayid, M., & Izadkhah, S. (2014). Mean inactivity time function, associated orderings and classes of life distributions. *IEEE Transactions on Reliability*, *63*(2), 593–602. https://doi.org/10.1109/TR.2014.2315954

Keller, A. Z., Kamath, A. R. R., & Perera, U. D. (1982). Reliability Analysis of CNC Machine Tools. *Reliability Engineering*, *3*(6), 449–473. https://doi.org/10.1016/0143-8174(82)90036-1

Khan, M. S., & King, R. (2012). Modified Inverse Weibull Distribution. *Journal of Statistics Applications & Probability*, *1*(2), 115–132. https://doi.org/10.12785/jsap/010204

Lee, S., Noh, Y., & Chung, Y. (2017). Inverted exponentiated Weibull distribution with applications to lifetime data. *Communications for Statistical Applications and Methods*, *24*(3), 227–240. https://doi.org/10.5351/csam.2017.24.3.227

Lin, Y. -J., Okasha, H. M., Basheer, A. M., & Lio, Y. L. (2023). Bayesian Estimation of Marshall Olkin Extended Inverse Weibull Under Progressive Type II Censoring. *Quality and Reliability Engineering International*, *39*(3), 931–957. https://doi.org/10.1002/qre.3270

Mahdavi, A., & Kundu, D. (2017). A New Method for Generating Distributions with an Application to Exponential Distribution. *Communications in Statistics - Theory and Methods*, *46*(13), 6543–6557. https://doi.org/10.1080/03610926.2015.1130839

Mubarak, A. E., & Al-Metwally, E. (2021). *A New Extension Exponential Distribution with Applications of COVID-19 Data*.

Mudholkar, G. S., & Kollia, G. D. (1994). Generalized Weibull Family: A Structural Analysis. *Communications in Statistics - Theory and Methods*, *23*(4), 1149–1171. https://doi.org/10.1080/03610929408831309

Nassar, M., Alzaatreh, A., Mead, M., & Abo-Kasem, O. (2017). Alpha Power Weibull Distribution: Properties and Applications. *Communications in Statistics - Theory and Methods*, *46*(20), 10236–10252. https://doi.org/10.1080/03610926.2016.1231816

Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M., & Afify, A. Z. (2019). The Marshall–Olkin alpha power family of distributions with applications. *Journal of Computational and Applied Mathematics*, *351*, 41–53. https://doi.org/10.1016/j.cam.2018.10.052

Okasha, H. M., Basheer, A. M., & Lio, Y. (2022). The E-Bayesian Methods for the Inverse Weibull Distribution Rate Parameter Based on Two Types of Error Loss Functions. *Mathematics*, *10*(24), 4826. https://doi.org/10.3390/math10244826

Okasha, H. M., Basheer, A. M., & El-Baz, A. H. (2021). Marshall–Olkin Extended Inverse Weibull Distribution: Different Methods of Estimations. *Annals of Data Science*, *8*(4), 769–784. https://doi.org/10.1007/s40745-020-00299-5

Okasha, H. M., El-Baz, A. H., & Basheer, A. M. (2020a). Bayesian Estimation of Marshall Olkin Extended Inverse Weibull Distribution Using MCMC Approach. *Journal of the Indian Society for Probability and Statistics*, *21*(1), 247–257. https://doi.org/10.1007/s41096-020-00082-y

Okasha, H. M., El-Baz, A. H., & Basheer, A. M. (2020b). On Marshall-Olkin Extended Inverse Weibull Distribution: Properties and Estimation Using Type-II Censoring Data. *Journal of Statistics Applications & Probability Letters*, *7*(1), 9–21. https://doi.org/10.18576/jsapl/070102

Okasha, H. M., El-Baz, A. H., Tarabia, A. M. K., & Basheer, A. M. (2017). Extended inverse Weibull distribution with reliability application. *Journal of the Egyptian Mathematical Society*, *25*(3), 343–349. https://doi.org/10.1016/j.joems.2017.02.006

Pararai, M., Warahena, G., & Oluyede, B. O. (2014). A New Class of Generalized Inverse Weibull Distribution with Applications. *Journal of Applied Mathematics and Bioinformatics*, *4*(2), 17–35.

Saboori, H., Barmalzan, G., & Ayat, S. M. (2020). Generalized Modified Inverse Weibull Distribution: Its Properties and Applications. *Sankhya B*, *82*(2), 247–269. https://doi.org/10.1007/s13571-018-0182-1